MTHSC 4420 Advanced Mathematical Programming Test 1

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Problem 1

UPS has certain constraints on the size of a package that can be shopped at standard rates. The size is measured as follows:

- *The length of the longest side is the length of the package. The lengths of the other two sides are the width and height, but it doesn't matter which is which. The girth is twice the sum of the width and the height. The size of the package is length plus girth.*
- *The maximum length is 108 inches. The maximum size of the package is 165 inches.*

a.)*Formulate a mathematical program to find the maximum volume a shippable box can hold and the dimensions (length, width, and height) of a shippable box of maximum volume.*

Let:
volume = length $*$ width $*$ height
girth = 2(width+height)
size =length+girth

max volume
s.t. length ≤ 108
size ≤ 165

b.) *Code your model in AMPL and solve it using the MINOS solver.*

AMPL Code
AMPL Mod Code

```
var length default 1 ;
var width default 1;
var height default 1;

maximize volume: length * width * height;

subject to length_longer_height: length >= height;

subject to length_longer_width: length >= width;

subject to length_restriction: length <= 108;

subject to size_restriction: length + 2 * (width + height) <= 165;
```

AMPL Code
AMPL Run Code

```
reset;
model prob1mod.mod;
option solver minos;
solve;
display volume;
display length;
display width;
display height;
```

AMPL Solution

```
ampl: include run.run;
MINOS 5.51: optimal solution found.
7 iterations, objective 41593.75
Nonlin evals: obj = 16, grad = 15.
volume = 41593.7
```

```
length = 55

width = 27.5

height = 27.5
```

c.) *AMPL can tell you the values of constraint slacks and Lagrange multipliers. Which constraints are tight? What are the associated multipliers?*

```
length_restriction.slack = 53
size_restriction.slack = 0
length_restriction.dual = 0
size_restriction.dual = 756.25
```

From AMPL, we know that the slacks on the length constraint is 53. The slack on the size constraint is 0.

The lagrange multipliers for the length constraint is 0 and the langrange multiplier of the size constraint is 756.25.

Therefore, the size constraint is tight since the lagrange multiplier is nonzero and the length constraint is not tight since the lagrange multiplier is zero.

d.) *Write down the KKT optimality conditions for this problem. Verify that the proposed solution satisfies the KKT conditions.*

The KKT conditions consist of the primal constraints, the gradient equation, the complementary slackness, and the sign restrictions.

The primal constraints are:
length ≤ 108
size ≤ 165

The gradients are:
$f(\text{length,width,height}) = \text{length} * \text{width} * \text{height}$
$g_1(\text{length, width, height}) = \text{length } -108 = 0$
$g_2(\text{length, width, height}) = \text{length } +2(\text{width + height}) - 165 = 0$
$g_3(\text{length, width, height}) = \text{length} \geq 0$
$g_4(\text{length, width, height}) = \text{width} \geq 0$
$g_5(\text{length, width, height}) = \text{height} \geq 0$

The gradients are:
$\nabla f(\text{length,width,height}) = (\text{width}*\text{height},\text{length}*\text{height},\text{length}*\text{width})$
$\nabla g_1(\text{length, width, height}) = (1, 0, 0)$
$\nabla g_2(\text{length, width, height}) = (1, 2, 2)$
$\nabla g_3(\text{length, width, height}) = (1, 0, 0)$
$\nabla g_4(\text{length, width, height}) = (0, 1, 0)$
$\nabla g_5(\text{length, width, height}) = (0, 0, 1)$

Therefore, the gradient equation is:
$(\text{width}*\text{height},\text{length}*\text{height},\text{length}*\text{width}) = (1, 0, 0)\lambda_1+(1, 2, 2)\lambda_2+(1, 0, 0)\mu_1+(0, 1, 0)\mu_2 + (0, 0, 1)\mu_3$

The complementary slackness conditions are:
$\lambda_1(\text{length } -108) = 0$
$\lambda_2(\text{length } +2(\text{width } + \text{ height}) - 165) = 0$
$\mu_1(0- \text{ length}) = 0$
$\mu_2(0- \text{ width}) = 0$
$\mu_3(0- \text{ height}) = 0$

The sign restrictions are:
$\lambda_1, \lambda_2 \geq 0$
$\mu_1, \mu_2, \mu_3 \leq 0$

e.)*Predict the effect of increasing the size limit by one inch. Solve the revised model with AMPL and compare the new objective value with your prediction. What accounts for the difference?*

I predict the volume to increase and the dimensions to also increase, but the number of iterations to remain the same.

Revised AMPL Solution

```
ampl: include run.run;
MINOS 5.51: optimal solution found.
7 iterations, objective 42354.59259
Nonlin evals: obj = 16, grad = 15.
volume = 42354.6

length = 55.3333

width = 27.6667

height = 27.6667
```

The original opbjective value was a volume of 41593.75. The new objective value is 42354.6. This represents an increase of 760.85, which is significant. The difference is accounted for by the relaxation of the size constraint. Since size deals with all three variables, the increase is distrubted among the length, width, and height. The increase in the width and the height is more pronounced since in the size equation the height and width variables have a scalar multiple of 2.

f.) *How should each additional inch of size be distributed between length, width, and height to maintain maximum volume?*

As we see in part (d), the extra inch should be distributed as follows: $\frac{1}{3}$ to length and $\frac{2}{3}$ to height and width, since the size constraint has a coefficient of 2 for each of those variables.

Problem 2

Consider the NLP:
maximize $14x - x^2 + 6y - y^2 + 7$
$s.t x + y \leq 2$
$x + 2y \leq 3$
$x \geq 0$
$y \geq 0$

a.) *Verify that the objective is concave and the feasible region is convex.*

$$\mathbf{H} = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$$

Since, the determinants of the principal submatrices are -2, and 0, the Hessian is negative semidefinite and the objective function is concave.

The constraints are both linear and thus convex. Therefore the feasible region is convex.

b.) *Write the Lagrangian function for this program.*

$L(x, y) = 14x - x^2 + 6y - y^2 + 7 + \lambda_1(2 - x - y) + \lambda_2(3 - x - 2y) + \mu_1(-x) + \mu_2(-y)$

c.) *Write the KKT conditions.*

The KKT conditions consist of the primal constraints, the gradient equation, the complementary slackness, and the sign restrictions.

The primal constraints are:
$x + y \leq 2$
$x + 2y \leq 3$
$x \geq 0$
$y \geq 0$

The gradients are:

$g_1(x) = x + y - 2$
$g_2(x, y) = x + 2y - 3$
$g_3(x, y) = x$
$g_4(x, y) = y$

$\nabla f(x, y) = (-2(x - 7), 6 - 2y)$
$\nabla g_1(x, y) = (1, 1)$
$\nabla g_2(x, y) = (1, 2)$
$\nabla g_3(x, y) = (1, 0)$
$\nabla g_4(x, y) = (0, 1)$

Therefore, the gradient equation is:
$(-2(x - 7), 6 - 2y) = (1, 1)\mu_1 + (1, 2)\mu_2 + (1, 0)\mu_3 + (0, 1)\mu_4$

The complementary slackness conditions are:
$\lambda_1(x + y - 2) = 0$
$\lambda_2(x + 2y - 3) = 0$
$\mu_1(0 - x) = 0$
$\mu_2(0 - y) = 0$

The sign restrictions are:
$\lambda_1, \lambda_2 \geq 0$
$\mu_1, \mu_2 \leq 0$

d.)*Find all KKT points. Which KKT points are feasible? Which feasible KKT point maximizes the objective? Which point corresponds to the unconstrained maximum?*

There are 4 constraints, so there are $2^4 = 16$ associated linear programs.

The 16 permutations are and the associated KKT points are:

Permutation	KKT Point	Objective
0000	NLB	10
0001	NLB	10
0010	NUB	10
0011	**(0,0)**	7
0100	NLB, NUB	10
0101	(3,0)	40
0110	**(0,1.5)**	13.75
0111	(3,0)	37
1000	NLB, NUB	10
1001	(2,0)	31
1010	(0,2)	15
1011	**(2,0)**	29
1100	**(1,1)**	25
1101	(3,0)	39
1110	**(0,1.5)**	14.25
1111	(3,0)	36

Where, NLB = No Lower Bound and NUB = No Upper Bound.

The points in bold are feasible. The point in blue is the feasible KKT point which maximizes the objective. The point in red is the KKT point that corresponds to the unconstrained maximum.