Integral Review

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Bernhard Riemann

- Bernhard Riemann (1826-1866) was an influential mathematician who studied under Gauss at the University of Gottingen.
- He made major contributions to mathematics including complex variables, number theory, and the foundations of geometry.
- We use Riemann’s definition of the definite integral.

Definite Integrals

- If a function *f* is defined for $a \le x \le b$, we divide the interval *[a,b]* into *n* subintervals of equal width $(\Delta = (b-a)/n)$
- Let *a* & *b* be the endpoints of these intervals.
- Let x_i^* be sample points in the intervals for all i = 1…n.
- Then the definite integral of *f* from *a* to *b* is

$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

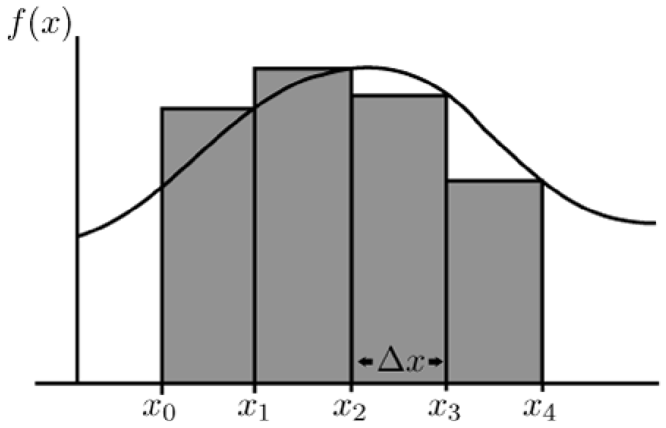
Integration Terms

- Integrand: $f(x)$
- Limits of Integration: a,b
- Upper Limit: b
- Lower Limit: a

$$\int_a^b f(x)dx$$

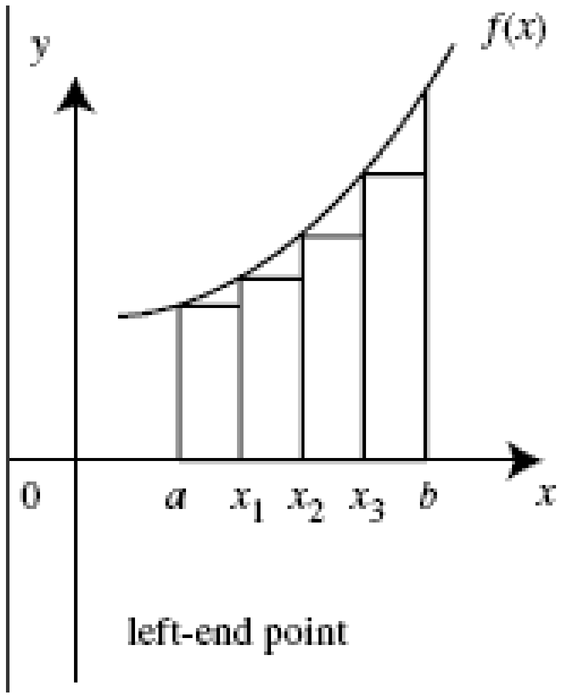
Riemann Sums

- The sum $\sum_{i=1}^{n} f(x_i^{\ *})\Delta x$ is called a **Riemann sum.**
- The definite integral of an integrable function can be approximated using a Riemann sum.
- If f is positive, then this approximation is interpreted as the sum of areas of approximating rectangles.



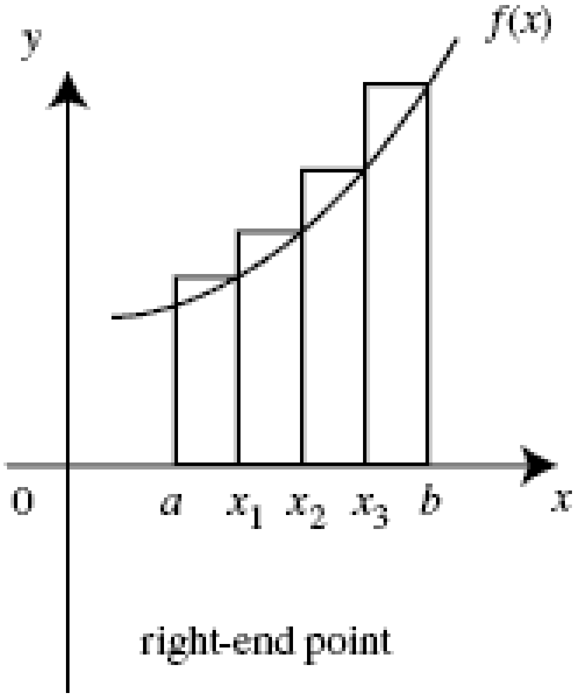
Left Riemann Sums

- For a left Riemann sum, the endpoints that are used to construct your ‘rectangles’ are *a*, x_1,…,*b-1*, but do not include *b* (the right endpoint).
- Left Riemann sums underestimate the area under the curve.



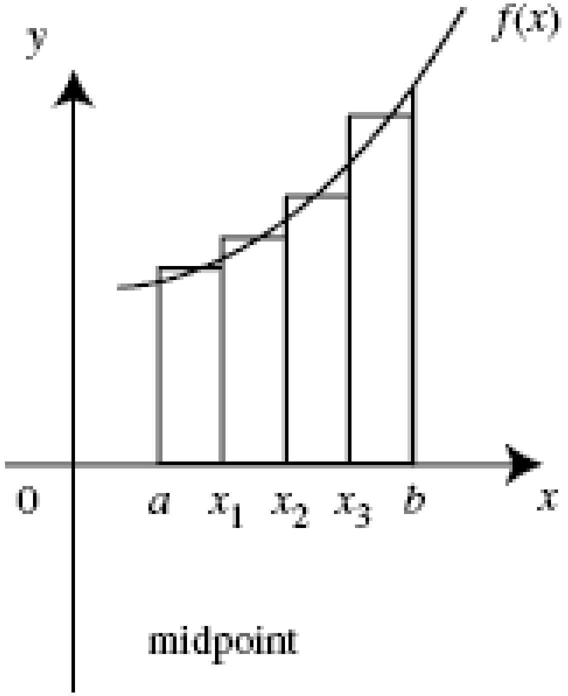
Right Riemann Sums

- For a right Riemann sum, the endpoints that are used to construct your ‘rectangles’ are $a+1$, $x_1,\ldots,b$, but do not include *a* (the left endpoint).
- Right Riemann sums over estimate the area under the curve.

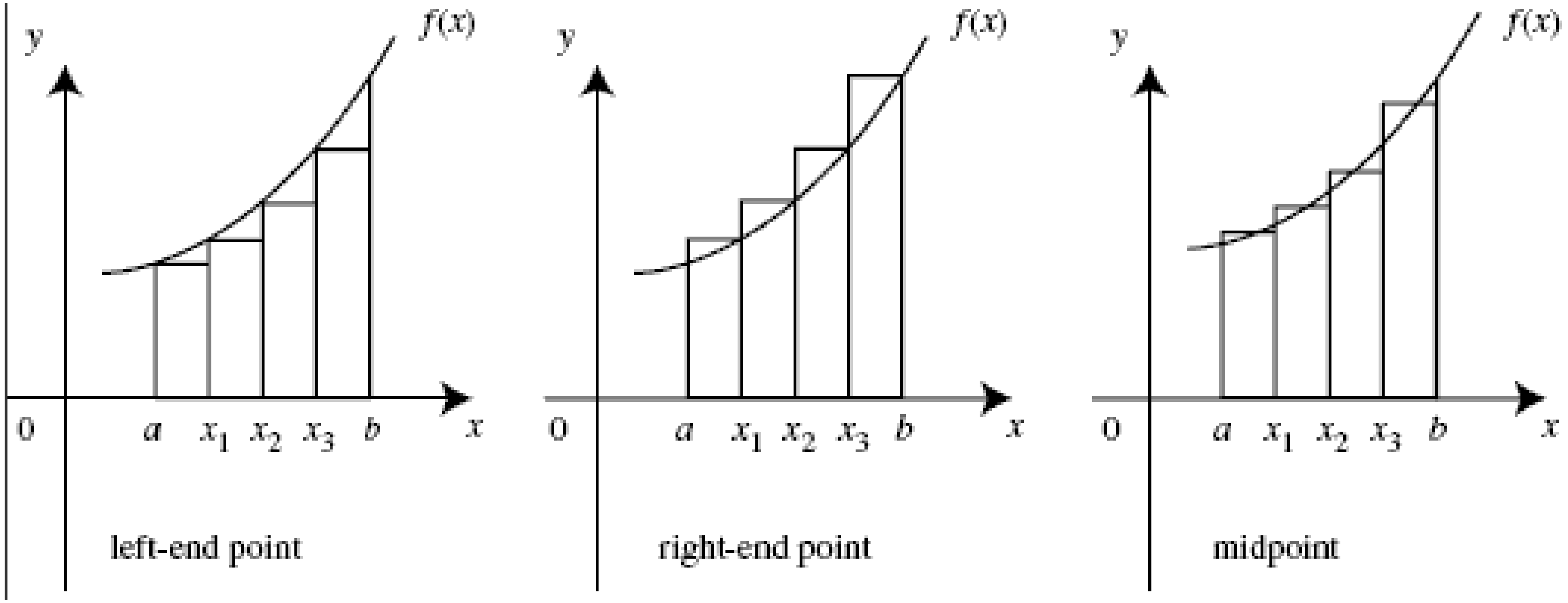


Mid Riemann Sums

- For a mid Riemann sum, the endpoints that are used to construct your 'rectangles' are $(a + x_1)/2$, $(x_2 + x_3)/2$ …,$(x_{b-1}+b)/2$
- Mid Riemann sums provide a balance between under (left) and over (right) estimating the area.

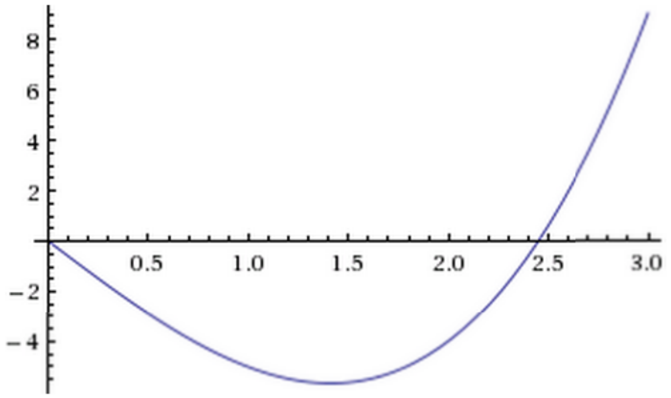


Left, Right, and Mid Points



Riemann Sum Example

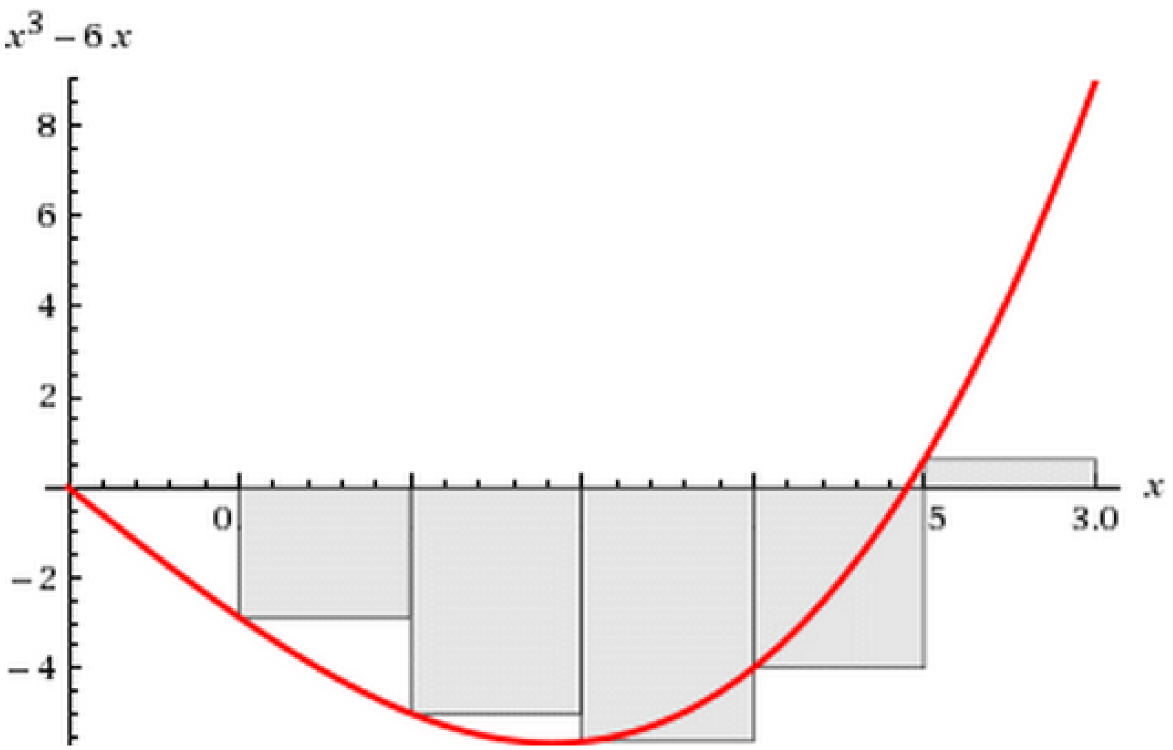
- Evaluate the Riemann sum for $f(x) = x^3 - 6x$ taking the sample points to be left endpoints and $a = 0$, $b = 3$, and $n = 6$.



Riemann Example: Find the Width

- First, we need to find the width of our rectangles.
- With *n* = *6*, we have

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{6} = \frac{1}{2}$$

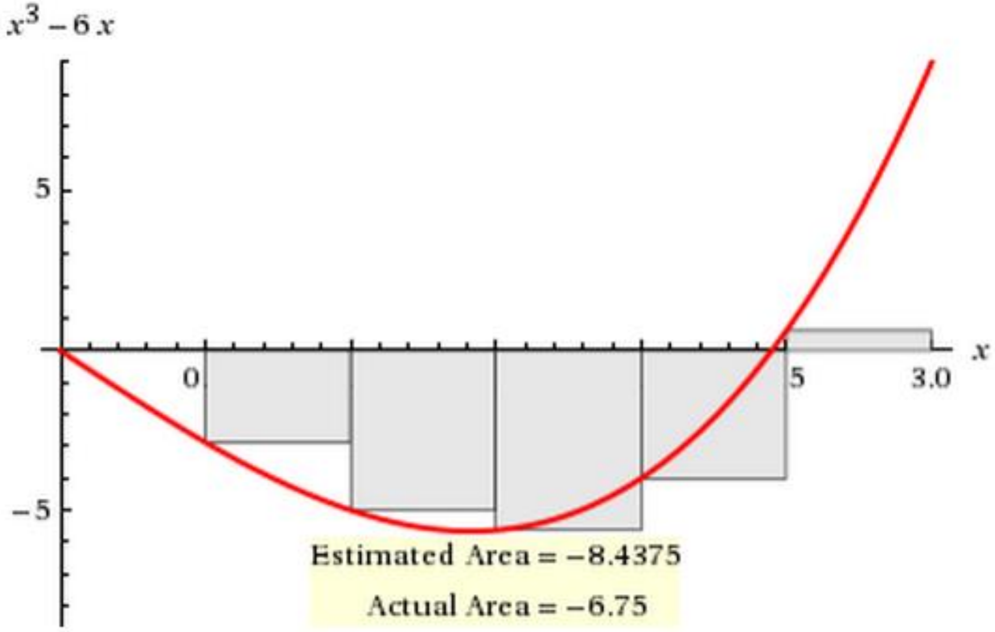


Riemann Example: Graph the Rectangles

Remember, we are looking for the left Riemann sum with *n=6* and a width of ½.

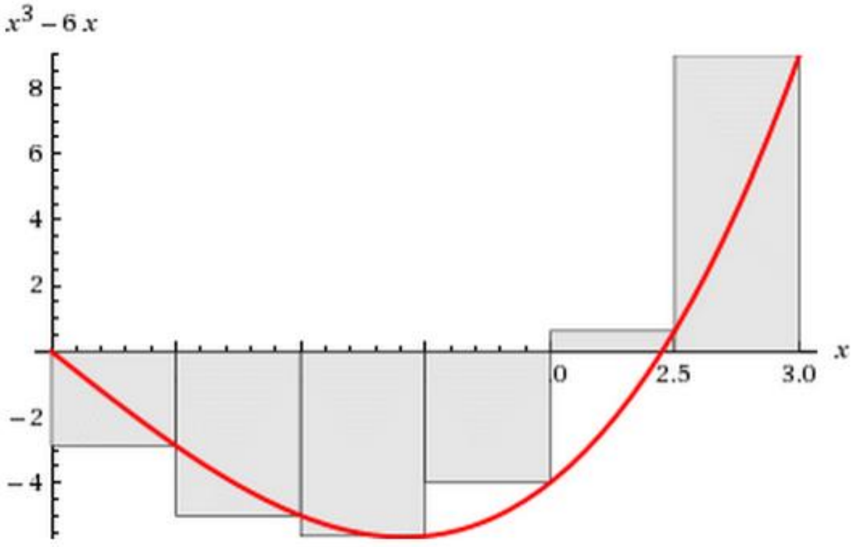
Riemann Sum: Final Area

- To calculate the Riemann sum, find the area of all 6 rectangles and add them together.

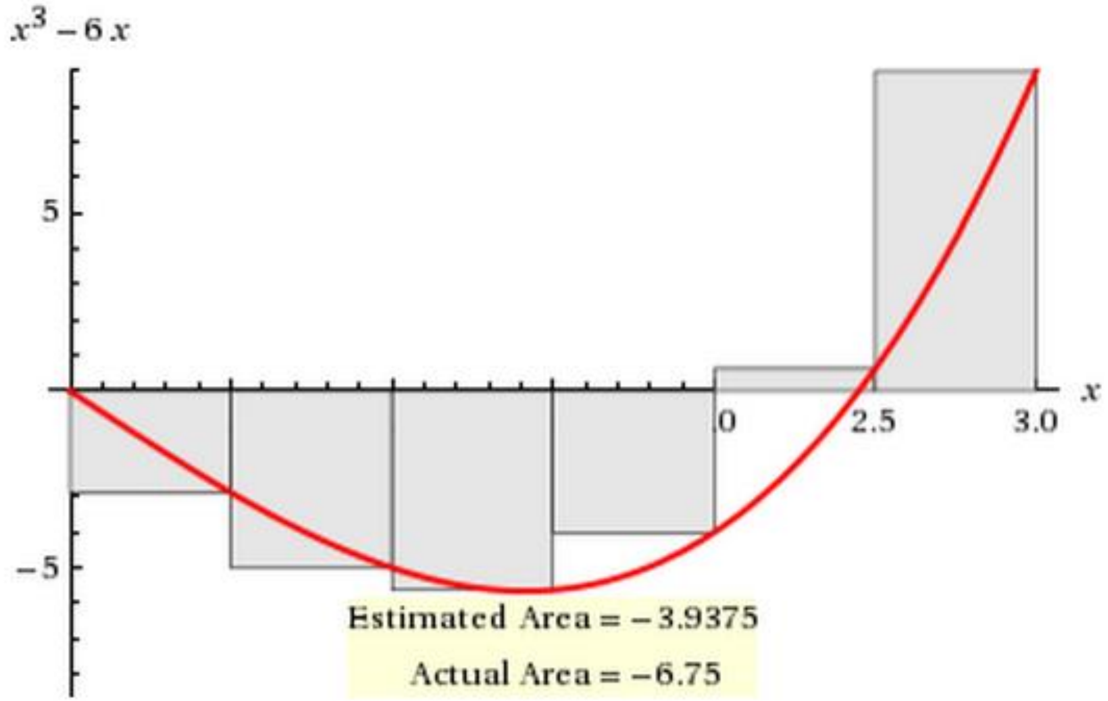


Riemann Sum: Right

- Now, let's look at the right Riemann sum.
- We will still have *n* = *6* and thus our width will remain the same.



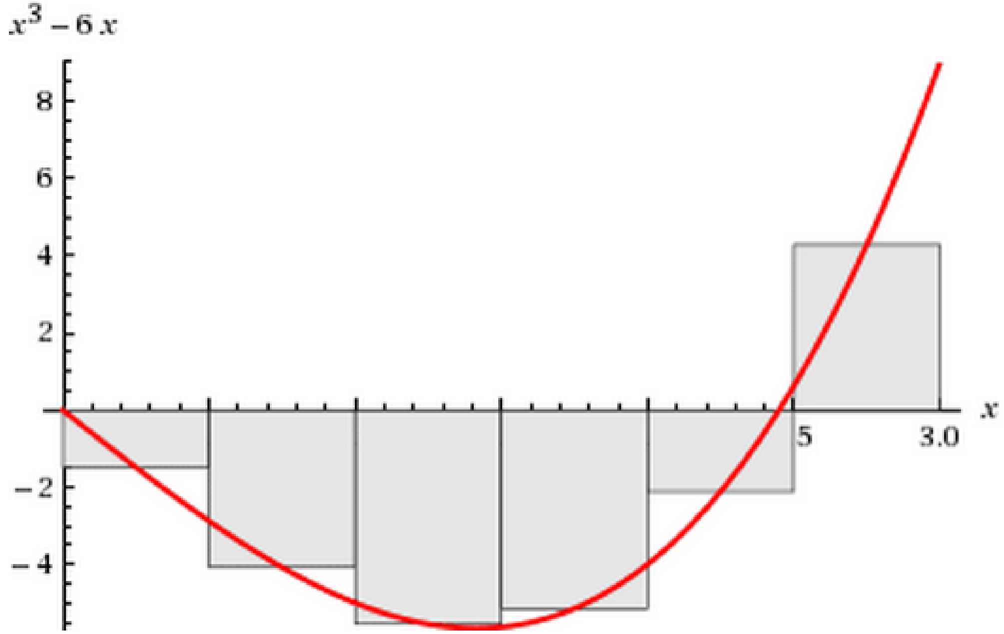
Riemann Sums: Right Area



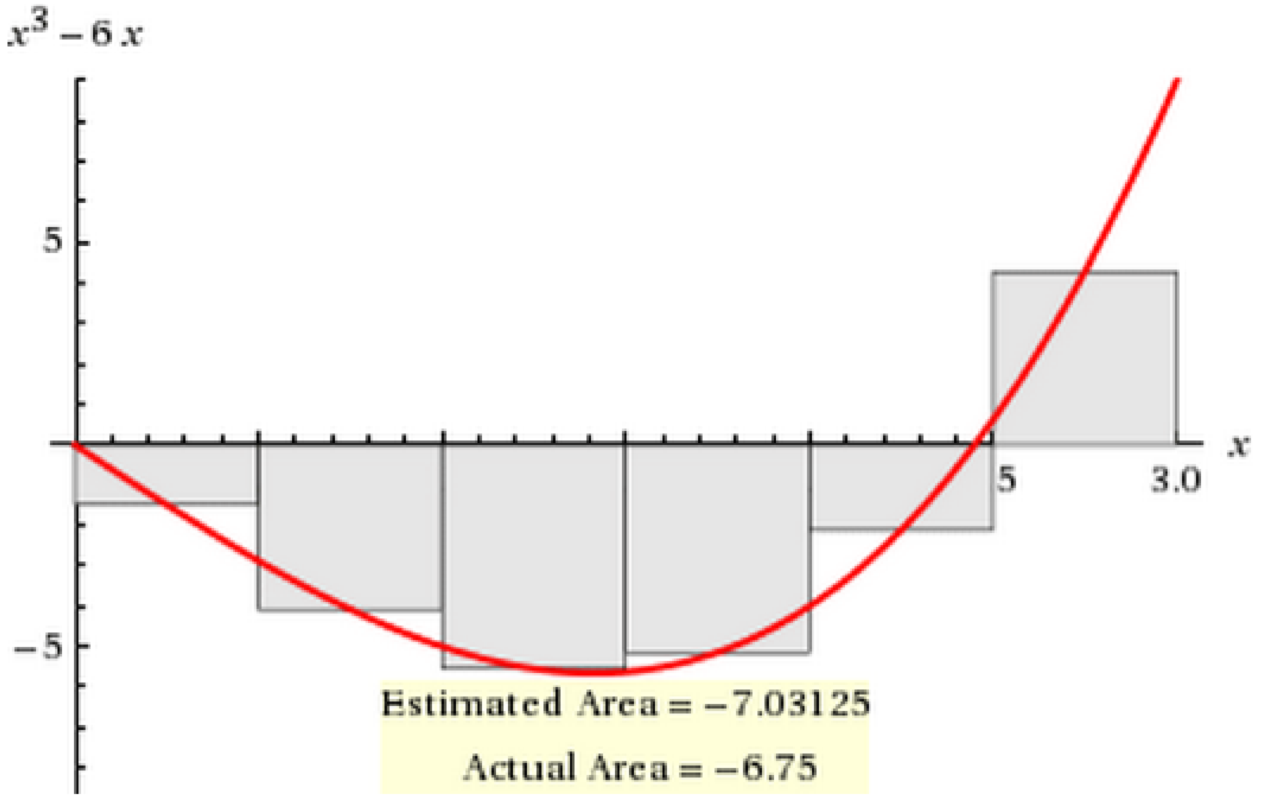
Is this a better estimate? Why or why not? Is there a better way?

Mid Riemann Sum

- Let’s try the Midpoint. We still have $n = 6$ and a width of ½.



Mid Riemann Sum Area



This is a better estimate than the left and right Riemann sums separately.
Midpoints often give a better estimate of the area under the curve.

First Fundamental Theorem of Calculus

- The fundamental theorem of calculus is a theorem that links the concept of the derivative of a function with the concept of the function's integral.
- The **first fundamental theorem of calculus** is that the definite integration of a function is related to its anti derivative.

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Example 1

- Use the fundamental theorem of calculus to evaluate

$$\int_0^1 x^3 \text{dx}$$

- First, we need to find the antiderivative of $f(x) = x^3$.

$$F(x) = \int x^3 dx = \frac{x^4}{4}$$

Example 1

- Now, we can apply the first fundamental theorem of calculus.

$$\int_0^1 x^3\,dx = F(1) - F(0) = \frac{1^4}{4} - \frac{0^4}{4} = \frac{1}{4}$$

Example 2

- Use the fundamental theorem of calculus to evaluate

$$\int_0^1 x^6 - 2x^3 + 5x \; dx$$

- First, we need to find the antiderivative of $f(x) = x^3$.

$$F(x) = \int x^6 - 2x^3 + 5x \, dx = \frac{x^7}{7} - \frac{2x^4}{4} + \frac{5x^2}{2} - 3x$$

Example 2

- Now, we can apply the first fundamental theorem of calculus.

$$\int_0^1 x^6 - 2x^3 + 5x \; dx = F(1) - F(0) =$$

$$= \frac{1^7}{7} - \frac{2 * 1^4}{4} + \frac{5 * 1^2}{2} - 3 * 1 - \left(\frac{0^7}{7} - \frac{2 * 0^4}{4} + \frac{5 * 0^2}{2} - 3 * 0\right)$$

$$= \frac{-6}{7}$$