Spring 2015, Math 111
Lab 2: Exploring the Limit

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Outline

What is a Limit?

In calculus, we are often interested in the values of $f(x)$ when x is very close to a number a, but not necessarily equal to a.

Definition

Given a function f, we say that the **limit** of f as x approaches a is L if, as x gets closer & closer to a, $f(x)$ gets closer and closer to L.

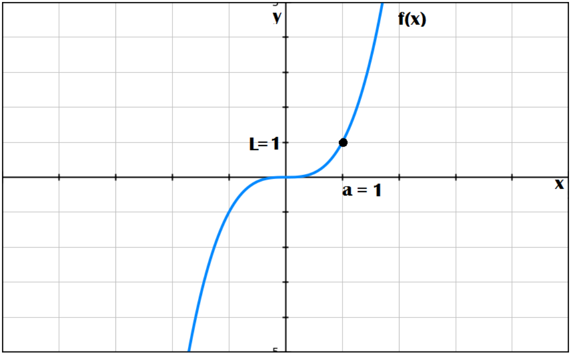
Symbolically, a limit is written as

$$\lim_{x \to a} f(x) = L$$

.

Example

Here we have a graph of the function $f(x) = x^3$.



We know that the $\lim_{x \to a} f(x) = L$. Therefore, $\lim_{x \to 1} x^3 = 1$

One Sided Limits

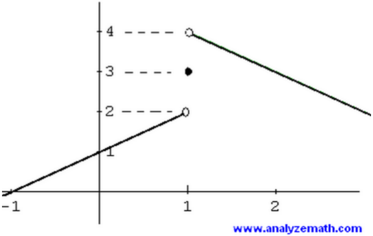
In the previous slide, we saw that the limit of the function was 1.

This is true because the function $f(x) = x^3$ is continuous and approached the same limit, L from both the left hand side and right hand side of the graph.

If the left-hand side and right-hand side limits do not equal, then the limit does not exist.

One Sided Limits

What does it look like if the limit does not exist?



Here we see that as the function approaches $a = 1$ from the left, the limit $L = 2$.
But, as the function approaches $a = 1$ from the right, the limit $L = 4$.

One Sided Limits

Definition

We say that the limit of $f(x)$ as x approaches a from the **left** is equal to L if, as x gets closer and closer to a, but **less than** a, $f(x)$ gets closer and closer to L. Thus,

$$\lim_{x \to a^-} f(x) = L$$

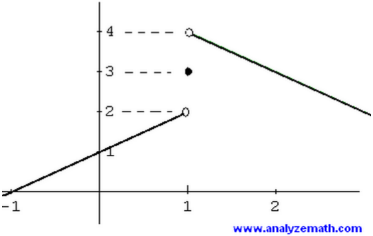
.

We say that the limit of $f(x)$ as x approaches a from the **right** is equal to L if, as x gets closer and closer to a, but **greater than** a, $f(x)$ gets closer and closer to L. Thus,

$$\lim_{x \to a^+} f(x) = L$$

One Sided Limits

Back to our previous example:



Left hand limit: $\lim_{x \to 1^-} f(x) = 2$

Right hand limit: $\lim_{x \to 1^+} f(x) = 4$

Limit Existence

It's possible for one-sided limits to exist and the limit not to exist. This gives us the following theorem.

Theorem

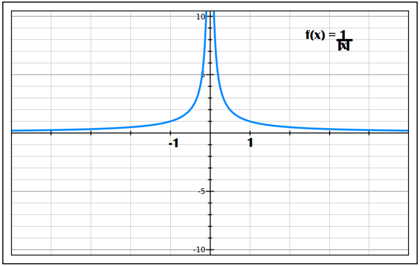
$\lim_{x \to a} f(x) = L$ *if and only if both* $\lim_{x \to a^-} f(x) = L$ *and* $\lim_{x \to a^+} f(x) = L$.

This means that for a limit to exist, the right and left hand limits must exist and must be equal.

Infinite Limits

An **infinite limit** occurs when the values of $f(x)$ tend towards either positive or negative infinity. Here we see that as the values of x tend towards 0, the values of $f(x)$ become arbitrarily large.

Graph of $f(x) = \frac{1}{|x|}$.



Infinite Limits

The table below corresponds to values of $f(x) = \frac{1}{|x|}$.

x	$f(x)$	x	$f(x)$
-0.5	**2**	0.5	**2**
-0.1	**10**	0.1	**10**
-0.01	**100**	0.01	**100**
-0.001	**1000**	0.001	**1000**

As we get closer and closer to $x = 0$, $f(x)$ increases.
However, $f(x)$ is not approaching a specific number L. Therefore, the limit does not exist. We denote this **infinite limit** as

$$\lim_{x \to 0} \frac{1}{|x|} = \infty$$

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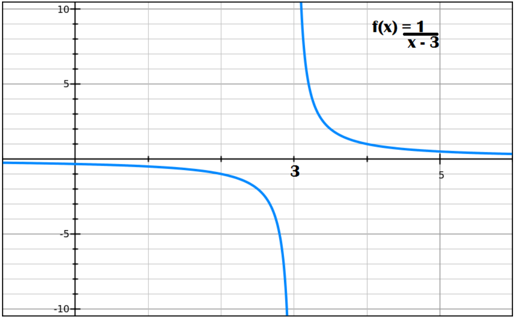
Infinite Limits

Now, let's look at a function $f(x) = \frac{1}{x-3}$, with different right and left hand limits.



Left-hand limit: $\lim_{x \to 3^-} \frac{1}{x-3} = -\infty$.

Right-hand limit: $\lim_{x \to 3^+} \frac{1}{x-3} = \infty$

Asymptotes

An **asymptote** is a line or curve that approaches a given curve or point arbitrarily closely, but does not 'touch' that point.

There are two main types of asymptotes: vertical and horizontal.

These asymptotes can be detected when limits exhibit specific behaviors.

Vertical Asymptotes

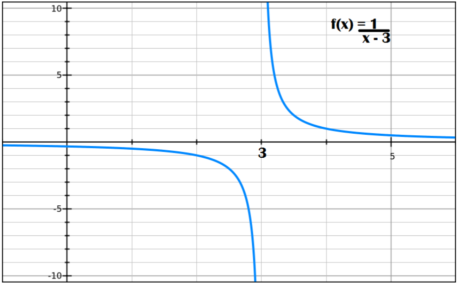
A **vertical asymptote** occurs when a function has an infinite limit as x approaches some a.

The line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = \infty$$
$$\lim_{x\to a} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = -\infty$$

Vertical Asymptotes

Recall our function $f(x) = \frac{1}{x-3}$



We see that $f(x)$ has a vertical asymptote at $x = 3$ since there is an infinite limit at 3, meaning $\lim_{x \to 3^+} f(x) = \infty$.

Note that $\lim_{x \to 3^-} f(x) = -\infty$.

Horizontal Asymptotes

A **horizontal asymptote** occurs when the limit of $f(x)$ approaches a specific value as x approaches ∞ or $-\infty$.

The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either:

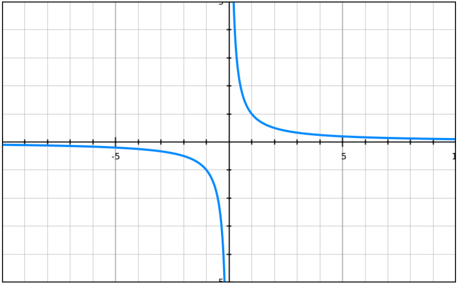
$$\lim_{x \to -\infty} f(x) = L \quad \text{or} \quad \lim_{x \to \infty} f(x) = L$$

These limits are referred to as limits at infinity.

Horizontal Asymptotes

We see that our $f(x) = \frac{1}{x}$ has the following limits at infinity:

$$\lim_{x \to -\infty} \frac{1}{x} = 0 \text{ and } \lim_{x \to \infty} \frac{1}{x} = 0$$



Therefore, $f(x)$ not only has a vertical asymptote at $x = 0$, it also has a horizontal asymptote at $y = 0$.

Asymptotic Behavior of Real Gases

Different gases have different equations of state, denoted $p(v)$.

Recall that the equation of state for an ideal gas is:

$$p(v) = \frac{nRT}{v}.$$

For the purposes of our lab, we will be considering one mole and thus $n = 1$.

R is the ideal gas constant. In this lab assignment, it is not necessary to substitute in a value for R.

Asymptotic Behavior of Real Gases

An example of a non-ideal equation of state is the Van der Waal:

$$p(v) = \frac{RT}{v - b} - \frac{a}{v^2}.$$

Here a is the measurement of attraction between particles and b is the volume excluded by a mole of particles. We will leave these as generic constants.

Asymptotic Behavior of Real Gases

One of the basic assumptions of thermodynamics is that as a gas becomes more dilute, it behaves more like an ideal gas.

A gas is said to be **dilute** when we increase the volume of the gas so as to "thin it out" while maintaining constant temperature.

Chemists capture this notion of diluting by writing

$$\lim_{v \to \infty} (v \times p(v)) = RT \text{ (DILUTE)} .$$

Where $v \times p(v) = RT$ is the equation of state of an ideal gas with fixed temperature.

Asymptotic Behavior of Real Gases

We can determine whether or not a gas satisfies the condition for dilute given the formula for the equation of state for any gas, $p(v)$.

For example, to determine whether or not the the Van der Waal satisfies the condition for dilute, we want to solve the following

$$\lim_{v \to \infty} (v \times p(v)) = \lim_{v \to \infty} (v \times (\frac{RT}{v-b} - \frac{a}{v^2}))$$

If this equals RT, then it satisfies the conditions for dilute.

Ending Notes

Lab 2 is due next Tuesday, 2/10/2015 in class.

- Proper limit notation is required throughout the lab. Points will be deducted accordingly.
- L'Hospital's Rule may not be used in this lab.
- Work must be shown to receive credit.
- Tutoring Monday night 5pm-8pm in Jones 112.