Spring 2015, Math 111
Lab 3: Exploring the Derivative

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Outline

What is a Derivative?

"The derivative is a way of measuring instantaneous change, which is a way of measuring change, such as finding the speed of a car when you only know its position." – Jennifer Ouellette, Author of *The Calculus Diaries*

What does this mean?

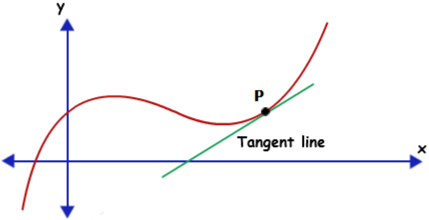
We can think of the derivative in several different ways:

- Slope of a function at a particular point.
- Slope of the tangent line to a function at a point.
- Limit of the slope of the secant line.
- Rate of change of a variable with respect to time.

Remember: The derivative is a function and varies from place to place (e.g. as you speed up, your car changes velocity).

Tangent Lines

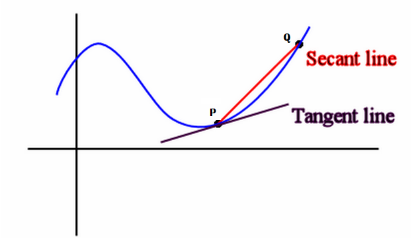
Let P be a point on the given curve.



A **tangent line** is a line that touches the given curve at P but doesn't cross over in an arbitrarily small neighborhood.

Secant Lines

Let Q be another point on the given curve.



A **secant line** is a line that intersects two or more points on a given curve.

Geometric Definition of Derivative

We can now define a derivative in terms of a tangent line.

Definition

Let f be a function and P be a point $(a, f(a))$. The derivative at point P is defined to be the slope of the tangent line to the graph of f at the point P. This is denoted **f'(a)**.

Slope of the Tangent Line

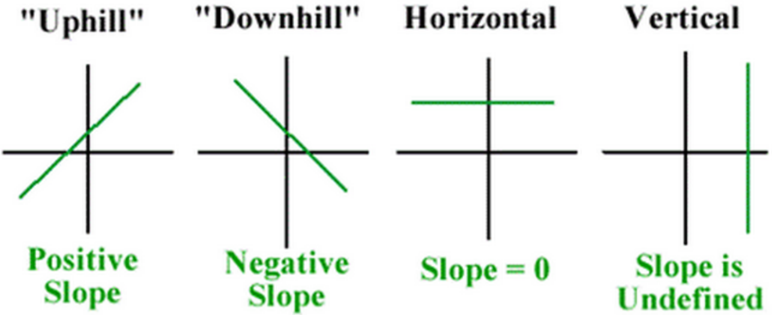
Remember, we know that the equation of a line is

$$y = mx + b$$

where m is the slope and b is the y-intercept.

However, to find the equation of the tangent line, we need two points. This is where we employ the concept of a secant line.

Positive, Negative, & Zero Slopes



Finding the Equation of the Tangent Line

Find the equation of the tangent line to $f(x) = 15 - 2x^2$ at $x_1 = 1$. First, evaluate $f(x)$ at $x = 1$.

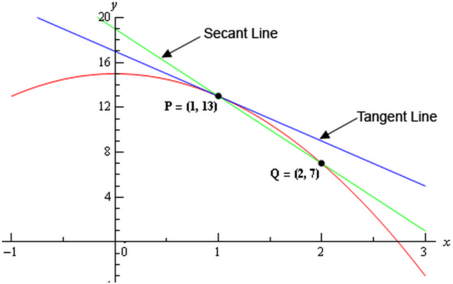
$$f(1) = 15 - 2(1)^2 = 13$$

Now we must construct a secant line by picking another point on the function $f(x)$. We will choose $x_2 = 2$

$$f(2) = 15 - 2(2)^2 = 7$$

Finding the Equation of the Tangent Line

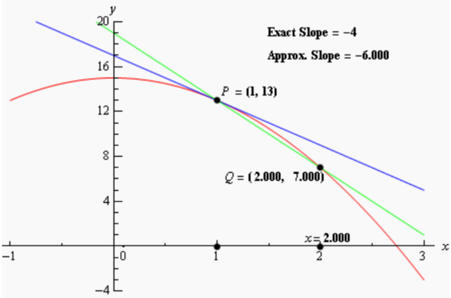
We see the two points graphed here.



To find the slope of the tangent line at $P = (1, 13)$ (i.e. the derivative), we must find the slope of the tangent line.

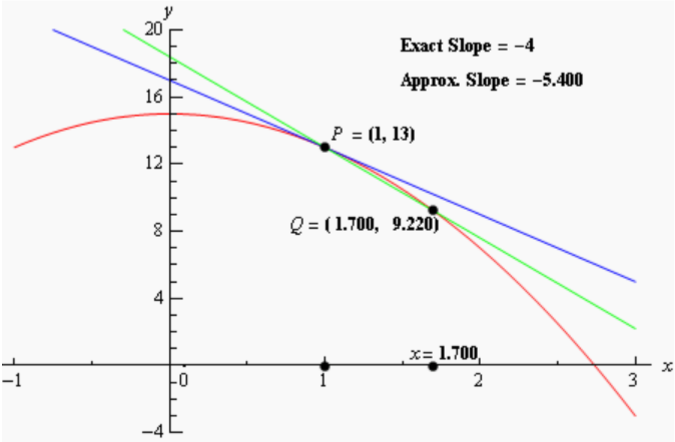
Finding the Equation of the Tangent Line

Slope of the secant line: $m = \frac{f(2)-f(1)}{2-1} = \frac{7-13}{1} = -6.$

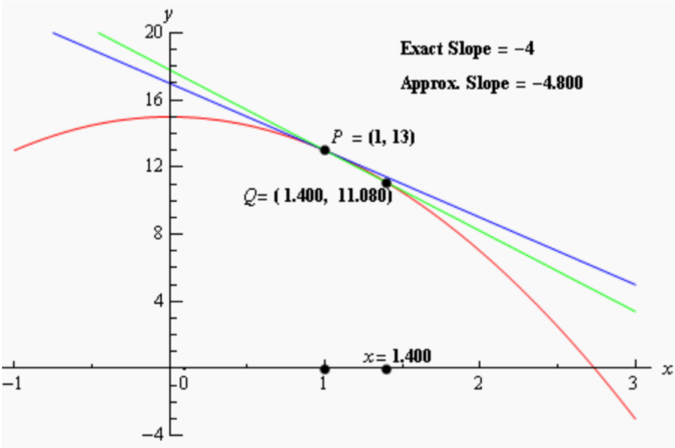


This is a good approximation of the slope of the tangent line, but it is not exact. How can we make it more exact?

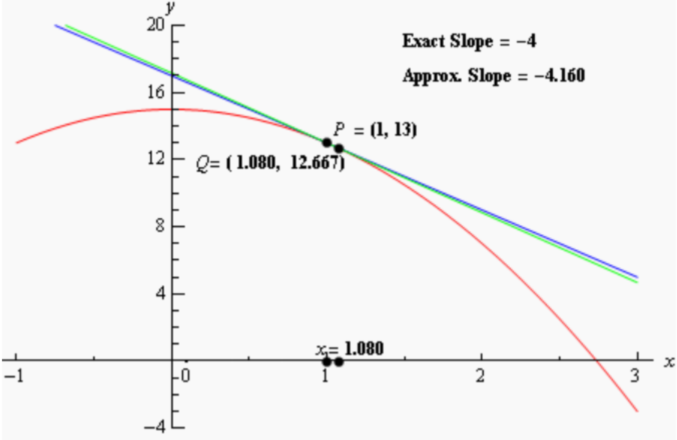
Finding the Equation of the Tangent Line



Finding the Equation of the Tangent Line



Finding the Equation of the Tangent Line



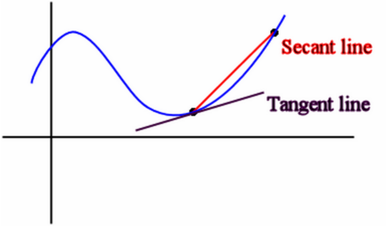
Finding the Equation of the Tangent Line

We see that as we move our second point Q closer and closer to P, we get closer and closer to the exact slope. Thus we can find the slope by looking at the limit of the function as Q approaches P. The final equation for the tangent line at point $P = (a, f(a))$ is

$$\begin{aligned} y &= f(a) + m(x - a) \\ &= 13 - 4(x - 1) \\ &= -4x + 17 \end{aligned}$$

Average Rate of Change

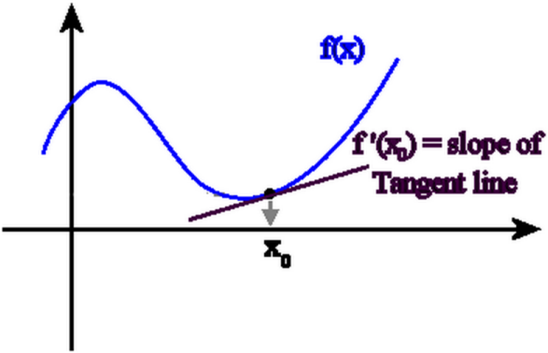
The secant line is a equivalent to the average rate of change for the given function.



$$\text{Average Rate of Change } = \frac{f(t_1) - f_(t_0)}{t_1 - t_0}$$

Instantaneous Rate of Change

The slope of the tangent line is a equivalent to the instantaneous rate of change for the given function.



Derivative of a Function at a Point

If we want to find the limit a specific x-value, $x = a$, we utilize the following definition.

Definition

The derivative of a function f at a number a, denoted $f`(a)$ is

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

if this limit exists.

Derivative of a Function at a Point

Find the derivative of $f(x) = x^2$ at $x = 2$.

$$
\begin{aligned}
f`(2) &= \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} \\
&= \lim_{x \to 2} \frac{x^2 - 4}{x - 2} \\
&= \lim_{x \to 2} \frac{(x+2)(x-2)}{x - 2} \\
&= \lim_{x \to 2} x + 2 \\
&= 4
\end{aligned}
$$

Derivative of a Function

If we want to find the derivate of a function for any x-value, we utilize the following definition.

Definition

The derivative of a function f is

$$f`(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if this limit exists.

Derivative of a Function

Find the derivative of $f(x) = x^2$.

$$\begin{aligned} f`(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h\to 0} \frac{h(2x+h)}{h} \\ &= \lim_{h\to 0} 2x + h \\ &= 2x \end{aligned}$$

Finding the Slope of the Tangent Line

We can use the definition of the derivative at a point to find the slope of the tangent line at a point using the following formula

$$m_{\tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Finding the Slope of the Tangent Line

Let $f(x) = x^2$. Find the slope of tangent line to $f(x)$ at $x = 3$.

$$\begin{aligned} m_{tan} &= \lim_{x\to 3} \frac{f(x) - f(3)}{x - 3} \\ &= \lim_{x\to 3} \frac{x^2 - 3^2}{x - 3} \\ &= \lim_{x\to 3} \frac{(x+3)(x-3)}{x - 3} \\ &= \lim_{x\to 3} x + 3 \\ &= 6 \end{aligned}$$

Internal Versus External Pressure of a Gas

This week we will be computing the rate of change of the external pressure of gases and using this to calculate the internal pressure.

Definition

For any gas, the **external pressure** is the pressure exerted by the gas on the walls of its container and is denoted as

$$p(T)$$

. The rate of change of the external pressure is its derivative:

$$p`(T).$$

Internal Pressure

For any gas, the **internal pressure** can be computed as the difference

$$T \times p`(T) - p(T).$$

Remember, *for this lab*, T is the temperature of the gas, p is the pressure, and v is the volume. Both T and p are variables while v is fixed (not a variable).

Internal Pressure

For the made-up equation of state: $p(T) = \frac{RT}{v-b} - \frac{1}{v-a}$, calculate the internal pressure.

$$p`(T) = \frac{R}{v-b}$$

.

$$\begin{aligned} T \times p`(T) - p(T) &= T * (\frac{R}{v-b}) - (\frac{RT}{v-b} - \frac{1}{v-a}) \\ &= \frac{RT}{v-b} - \frac{RT}{v-b} + \frac{1}{v-a} \\ &= \frac{1}{v-a} \end{aligned}$$

Ending Notes

Lab 3 is due next Tuesday, 2/17/2015 in class.

- Proper limit notation is required throughout the lab.
- Work must be shown to receive credit.
- Leave R as a constant.
- Tutoring Monday night 5pm-8pm in Jones 112.

These slides are available at www.meganrosebryant.com.