Spring 2015, Math 111
Lab 5: Related Rates

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Learning Objectives

Today, we will be looking at applications of implicit derivation in related rates problems.

Learning Objectives:

- Understand the concept of Related Rates
- Understand steps for solving Related Rates Problems
- Complete example problems

What are Related Rates?

A **Related Rate** problem is a one involving the derivation of a rate at which a quantity changes by relating that quantity to other quantities whose rates of change are known.

We will be considering rates of change with respect to time.

Chain Rule

Recall that if g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \times g'(x)$$

This can also be written as

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Chain Rule

Given the following function, find the derivative.

$$F(x) = sin(x^2)$$

Employ the chain rule with $f(u) = \sin(u)$ and $u = g(x) = x^2$.

$$\begin{aligned} F'(x) &= f'(g(x)) \times g'(x) \\ &= \sin(x^2) \times 2x \end{aligned}$$

Implicit Differentiation

Implicit Differentiation is the process of finding the derivative of a dependent variable in an implicit function by differentiating each term separately, by expressing the derivative of the dependent variable as a symbol, and by solving the resulting expression for the symbol.

What does this mean??

Implicit Differentiation

We are given the following function of a circle:

$$x^2 + y^2 = 36$$

We want to find $\frac{dy}{dx}$, which is the rate of change of x with respect to y. How can we accomplish this?

With *implicit differentiation*!

Implicit Differentiation

First, we need to differentiate both sides of the equation with respect to x.

$$x^2 + y^2 = 36$$
$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(36)$$
$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(36)$$
$$2x + 2y\frac{dy}{dx} = 0$$

Where did the $\frac{dy}{dx}$ come from? Let's take a closer look.

Chain Rule in Implicit Differentiation

When we derive an equation implicitly, we are actually employing the chain rule.

$$\begin{aligned}
\frac{d}{dx}(y^2) &= \frac{d}{dx} \times \frac{dy}{dy}(y^2) \\
&= \frac{d}{dy} \times \frac{dy}{dx}(y^2) \\
&= \frac{d}{dy}(y^2)\frac{dy}{dx} \\
&= 2y\frac{dy}{dx}
\end{aligned}$$

Implicit Differentiation

We have found the derivative of $x^2 + y^2 = 36$ with respect to x is

$$2x + 2y\frac{dy}{dx} = 0$$

But, we're not done yet. We wanted to find $\frac{dy}{dx}$.

We can do this using basic algebra to isolate $\frac{dy}{dx}$.

Implicit Differentiation

Use algebra to isolate $\frac{dy}{dx}$.

$$\begin{aligned} 2x + 2y\frac{dy}{dx} &= 0 \\ 2y\frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= -\frac{x}{y} \end{aligned}$$

We now know that the rate of change of y with respect to x,

$$\frac{dy}{dx} = -\frac{x}{y}$$

Related Rates: Example Problem

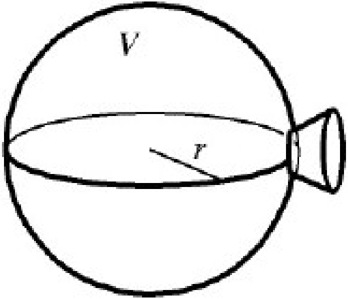
We now have all of the tools we need to find Related Rates!

Example 1: *Air is being pumped into the spherical balloon so that its volume increases at a rate of* $100cm^3/s$*. How fast is the radius of the balloon increasing when the diameter is* $50cm$*?*

How should we approach this problem?

Related Rates: Sketch the Problem

Let's begin by sketching the diagram and assigning variables.



V is the volume of the air. in the balloon and r is the radius.
Recall that the radius is half the diameter.

Related Rates: State the Given

Now, we want to identify which information is *given* in the problem statement.

We know that the rate of increase of the volume of air with respect to time is $100cm^3/s$.

Meaning, we are given that

$$\frac{dV}{dt} = 100cm^3/s$$

Related Rates: State the Unknown

Now that we have identified the *given* values, let's identify the *unknown* variable that we want to find.

We are asked to find the rate of increase of the radius when the diameter is 50cm (e.g. when radius is $\frac{50}{2} = 25$).

Meaning, we want to find

$$\frac{dr}{dt} \text{ when } r = 25cm$$

Related Rates: Linking Equation

In order to find the *unknown*, we need to define a **linking equation** that *relates* our identified variables V and r.

There is a natural choice here: the equation for the volume of a sphere.

$$V = \frac{4}{3}\pi r^3$$

Related Rates: Differentiate with Respect to Time

Now that we have our linking equation, we want to differentiate with respect to time. This will allow us to find our *unknown* quantity. Recall the linking equation: $V = \frac{4}{3}\pi r^3$.

$$\frac{d}{dt}(V) = \frac{d}{dt}(\frac{4}{3}\pi r^3)$$
$$\frac{d}{dt} \times \frac{dV}{dV}(V) = \frac{d}{dt} \times \frac{dr}{dr}(\frac{4}{3}\pi r^3)$$
$$\frac{d}{dV}(V)\frac{dV}{dt} = \frac{d}{dr}(\frac{4}{3}\pi r^3)\frac{dr}{dt}$$
$$(1)\frac{dV}{dt} = 3 * (\frac{4}{3}\pi r^2)\frac{dr}{dt}$$
$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Related Rates: Solve for Unknown

Now that we have differentiated with respect to time, we can use basic algebra to solve for our *unknown* variable.

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$
$$\Longrightarrow \frac{dr}{dt} = \frac{1}{4\pi r^2} \times \frac{dV}{dT}$$

Related Rates: Substitute in Given

We are now ready to substitute in our *given* values of $r = 25$ and $\frac{dV}{dt} = 100$.

$$\frac{dr}{dt} = \frac{1}{4\pi r^2} \times \frac{dV}{dT}$$
$$= \frac{1}{4\pi 25^2} \times 100$$
$$= \frac{1}{25\pi} cm/s$$

Therefore, we know that the rate of increase of the radius of the balloon with respect to time is $\frac{1}{25\pi} cm/s$.

Steps to Solve Related Rates

1. **Sketch**
2. Assign **variables** to quantities that change with time.
3. State the **given** information.
4. State the **unknown** variables/quantities that need to be found.
5. Define the **linking equation** that relates the variables/quantities.
6. **Differentiate** with respect to time.
7. **Substitute** the *given* values into the derivative.
8. **Solve** for the desired *unknown quantity*.

Hints for Lab 5

- For Exercise 2, the formula for the total volume of the water is the sum of two volume formulas ($V_{\text{Total}} = V_{\text{Cube}} + V_{\text{Triangular Solid}}$).
- For Problem 3 and Problem 4, create an x, y table of values to help express y in terms of x.

Ending Notes

Lab 4 is due Tuesday after Spring Break, 3/16/2015 in class.

- Proper derivative notation is required.
- Proper unit notation is required.
- Work must be shown to receive credit.
- Tutoring available Monday night 5pm-8pm in Jones 112.

These slides are available at www.meganrosebryant.com.