Spring 2015, Math 111
Lab 6: Linear Approximation

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Learning Objectives

Today, we will be looking at applications of linear approximation.

Learning Objectives:

- Derive a LA for a function at a particular value
- Interpret the LA graphically
- Use the LA of a function to estimate numerical values

What is Linear Approximation?

Linear approximation is the process of finding the equation of a line that is the closest estimate of a function for a given value of x.

It is sometimes called 'tangent line approximation'.

Applications of Linear Approximations

Linear Approximation has applications in:

- estimating derivatives
- trigonometric functions
- periods of oscillating functions
- Euler's method
- optics
- electrical resistivity

Estimating the Value of the Derivative

In Lab 3, the divided difference (slope formula)

$$\frac{f(x)-f(a)}{x-a}, x \neq a$$

was used to determine the value of the derivative

$$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

This is actually an approximation!

Estimating the Value of the Derivative

The derivative formula can be expressed as an approximation

$$f'(a) \approx \frac{f(x) - f(a)}{x - a}$$

which gives a reasonably good estimate for the derivative as long as $x - a$ is not *too* large.

It is commonly used when the limit calculation is too difficult or when only numerical data is available for the function in question.

Linear Approximations

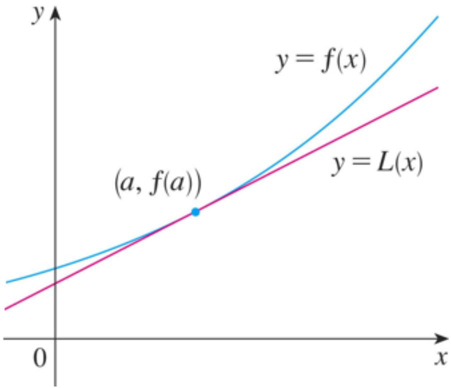
Linear approximations are used when it is easy to calculate a given value $f(a)$ of a function, but difficult or impossible to compute nearby values of f.

The formula is generalized to

$$f(x) \approx L(x) = f(a) + (x - a) \times f'(a)$$

where $L(x)$ is the linear approximation of f at a.

Linear Approximation Formula



$$f(x) \approx L(x) = f(a) + (x - a) \times f'(a)$$

Making and Using Linear Formulas

Trigonometric functions are useful in a wide variety of applications, but difficult to manipulate.

If a trigonometric function has a derivative at point a, linear approximation can be used as an easier-to-find replacement.

Example 1. Let $f(x) = \sin(x)$. Find the linear approximation to $f(x)$ at $x = 0$

Example 1

Recall:

$$f(x) \approx L(x) = f(a) + (x - a) \times f'(a)$$

What do we know?

$$a = 0$$
$$f(x) = \sin(x)$$
$$f'(x) = \cos(x)$$
$$\sin(a) = \sin(0) = 0$$
$$\cos(a) = \cos(0) = 1$$

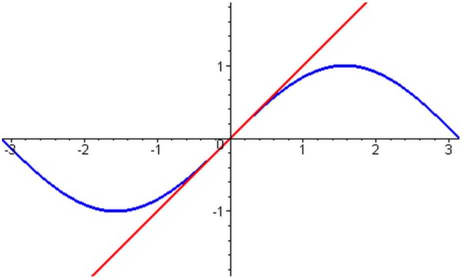
Example 1

Now, plug these values into our equation for the linear approximation

$$
\begin{aligned}
f(x) \approx L(x) &= f(a) + (x - a) \times f'(a) \\
\sin(x) \approx L(x) &= f(0) + (x - 0) \times f'(0) \\
&= 0 + (x) \times 1 \\
&= x
\end{aligned}
$$

Example 1

The graph of the linear approximation, $L(x)$, is a straight line tangent to the graph of $f(x) = \sin(x)$ at $(0, 0)$



Example 1

Let's see how accurate our linear approximation is by approximating the function value at 0.001.

$$L(x) = x$$
$$L(0.001) = 0.001$$

A calculator tells us that $\sin(0.001) = 0.001$, which is what the linear approximation returned. Therefore, the linear approximation accurately approximated the function value at 0.001.

Appropriate Base Values

The linear approximation of $\sin(0.001)$ was very accurate because the base value, $a = 0$, was very close to the point that was being estimated.

If the base value is too far away, the linear approximation will be more inaccurate.

Let's see what happens if we try to approximate $\sin(0.001)$ with a base value,a, of 2.

Appropriate Base Values

Approximate $\sin(0.001)$ with $a = 2$.

$$\begin{aligned} a &= 2 \\ f(x) &= \sin(x) \\ f'(x) &= \cos(x) \\ \sin(a) &= \sin(2) \approx 0.909 \\ \cos(a) &= \cos(2) \approx -0.416 \end{aligned}$$

Appropriate Base Values

Plugging these values in gives the following linear approximation.

$$\begin{aligned} f(x) \approx L(x) &= f(a) + (x-a) \times f'(a) \\ \sin(x) \approx L(x) &= f(2) + (x-2) \times f'(2) \\ &= \sin(2) + (x-2) \times \cos(2) \end{aligned}$$

Use this approximation to approximate sin(0.001).

$$\begin{aligned} L(0.001) &= \sin(2) + (0.001 - 2) \times \cos(2) \\ &\approx 1.74 \end{aligned}$$

Note that this value is much greater than the true value of 0.01. Therefore, always use a base value close to the value being approximated for an accurate result.

Estimating Numerical Values

Linear approximation can be used to get estimates for the change in value of some difficult function from a “standard” base-point to a nearby argument.

Example 2 Find $\sqrt[3]{0.97}$.

A calculator can tell us the exact answer quickly $(0.9\overline{8})$.

This allows us to test the accuracy of the linear approximation formula.

Example 2

To begin, determine an appropriate function $f(x)$ that models the numerical value to be estimated.

Since the desired value is a cubed root, use $f(x) = x^{\frac{1}{3}}$ as the function.

Next, to accurately estimate a numerical value, choose a value for a that is very close to the value being estimated.

To estimate $f(0.97) = \sqrt[3]{0.97}$, select 1 for the base value a.

Example 2

Thus, we have

$$
\begin{aligned}
a &= 8 \\
f(x) &= \sqrt[3]{x} \\
f'(x) &= \frac{1}{3\sqrt[3]{x^2}} \\
f(a) &= f(1) = \sqrt[3]{1} = 1 \\
f'(a) &= f'(1) = \frac{1}{3\sqrt[3]{1^2}} = \frac{1}{3}
\end{aligned}
$$

Example 2

Now, plug these values into the equation for the linear approximation

$$
\begin{aligned}
f(x) \approx L(x) &= f(a) + (x - a) \times f'(a) \\
\sqrt[3]{x} \approx L(x) &= f(1) + (x - 1) \times f'(1) \\
&= f(1) + (x - 1) \times f'(1) \\
&= 1 + (x - 1) \times \frac{1}{3}
\end{aligned}
$$

Example 2

The derived linear approximation

$$L(x) = 1 + (x - 1) \times \frac{1}{3}$$

can be used to estimate the value at our original point, 8.03

$$\begin{aligned} L(0.97) &= 1 + (1 - 0.97) \times \frac{1}{3} \\ &= 0.99 \end{aligned}$$

Note that we are very close to the exact value of $0.9\bar{8}$.

Hints for Lab 6

In problem 2(a), find the linear approximation for $f(T)$ and use T in place of x and T_0 in place of a.

Ending Notes

Lab 6 is due Tuesday, 3/23/2015 in class.

- Proper derivative notation is required.
- Work must be shown to receive credit.
- Tutoring available Monday night 5pm-8pm in Jones 112.

These slides are available at www.meganrosebryant.com.