Spring 2015, Math 111
Lab 8: Newton's Method

Megan Bryant

William and Mary

April 7, 2015

Learning Objectives

Today, we will be looking at applications of Newton's Method.

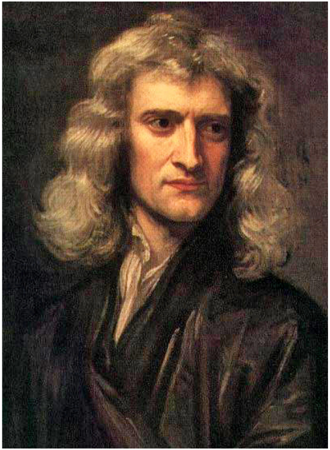
Learning Objectives:

- Estimate zeroes of a differentiable function using Newtons Method.
- Sketch the graphical interpretation of Newtons Method
- Recognize when Newtons Method fails and why it fails

Newton's Method: Historical Outline

Sir Isaac Newton (1643-1727) was an English physicist and mathematician who is widely recognised as one of the most influential scientists of all time and as a key figure in the scientific revolution

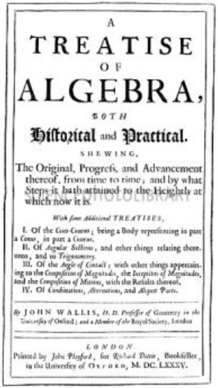
Newton's work has been said "to distinctly advance every branch of mathematics then studied".



Newton's Method: Historical Outline

The name "Newton's method" is derived from Isaac Newton's description of a special case of the method in *De analysi per aequationes numero terminorum infinitas* written in 1669.

It was first published in 1685 in *A Treatise of Algebra both Historical and Practical* by John Wallis.


A
TREATISE
OF
ALGEBRA,
BOTH
Historical and Practical.
SHEWING,
The Original, Progress, and Advancement thereof, from time to time; and by what Steps it hath attained to the Heighth at which now it is.

With some Additional TREATISES,

I. Of the Cono-Cuneus; being a Body representing in part a Conus, in part a Cuneus.
II. Of Angular Sections; and other things relating there-unto, and to Trigonometry.
III. Of the Angle of Contact; with other things appertaining to the Composition of Magnitudes, the Inceptives of Magnitudes, and the Composition of Motions, with the Results thereof.
IV. Of Combinations, Alternations, and Aliquot Parts.

By JOHN WALLIS, D. D. Professor of Geometry in the University of Oxford; and a Member of the Royal Society, London

LONDON
Printed by John Playford, for Richard Davis, Bookseller, in the University of OXFORD, M. DC. LXXXV.

Newton's Method: Intuition

Newton's Methond uses derivatives to estimate the real zero of a differentiable function $f(x)$.

These functions might be a polynomial, a trigonometric function, an exponential or logarithm function, or a complicated combination of these.

These equations may have one solution or many solutions. These x-intercepts are also called roots or the zeros of the function.

Newton's Method: Intuition

Even when $f(x)$ is a polynomial, the only simple solution formula is the quadratic formula, which solves equations for 2^{nd} degree polynomials.

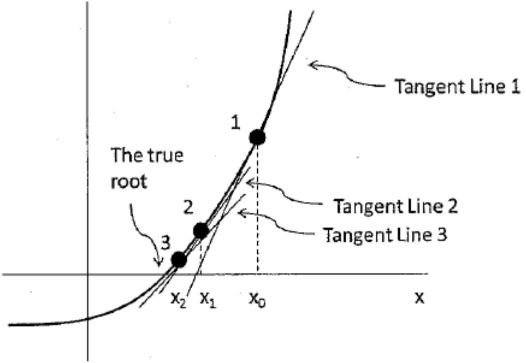
i.e. for $ax^2 + bx + c = 0$, the value of x is given by:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Functions other than polynomials rarely have formulas for the zeros. Newton's Method allows us to find these zeros under certain circumstances.

Newton's Method: Intuition

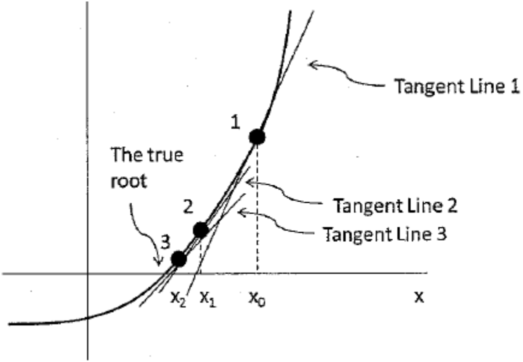
Geometrically, this formula begins with an initial guess, x_0, for a root.



If not, we can construct the tangent line to the graph of $f(x)$ at the point $(x_0, f(x_0))$.

Newton's Method: Intuition

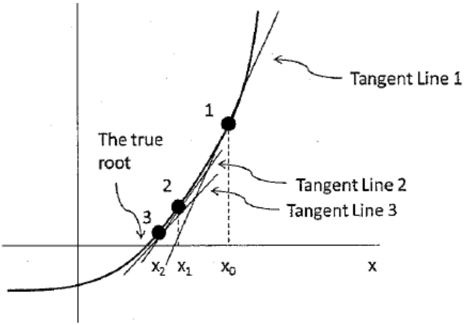
If x_0 is already a root, then $f(x_0) = 0$. If not, then we can graph the tangent line to $f(x)$ at the point $(x_0, f(x_0))$.



Then, because it is easy to find the x-intercept of this tangent line,

Newton's Method: Intuition

We can determine if this guess is an improved guess by comparing $f(x_0)$ with $f(x_1)$. If the latter is closer to 0 than the former, i.e. if $|f(x_1)| < |f(x_0)|$, we repeat the process to obtain an even better guess, x_2. Continue the process until we reach a guess x_n at which $f(x_n)$ is close enough to satisfy our needs.



Newton's Method: Formula

If we keep repeating this process, we obtain a sequence of approximations $x_1, x_2, x_3, \ldots$.

In general, if the nth approximation is x_n and $f'(x_n) =\neq 0$, then the next approximation is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

If $f'(x_{n+1})$ is 'close enough' to a value of 0, then stop and report x_{n+1}. Otherwise, recompute for the next value of x_{n+2}.

Newton's Method: Example 1

We will begin with an equation that Newton himself used to illustrate his method.

Starting with $x_1 = 2$, find the third approximation x_3 to the root of the equation $x^3 - 2x - 5 = 0$.

Newton's Method: Example 1

We will apply this method with

$$f(x) = x^3 - 2x - 5 \quad \text{and} \quad f'(x) = 3x^2 - 2$$

Thus we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$$

Newton's Method: Example 1

For $n = 1$,

$$\begin{aligned} x_{1+1} &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ x_2 &= x_1 - \frac{x_1^3 - 2x_1 - 5}{3x_1^2 - 2} \\ &= 2 - \frac{2^3 - 2*2 - 5}{3*2 - 2} \\ &= 2.1 \end{aligned}$$

Since we want to find the third approximation to the root, we need to continue for one more iteration.

Newton's Method: Example 1

For $n = 2$,

$$x_{2+1} = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = x_2 - \frac{x_2^3 - 2x_2 - 5}{3x_2^2 - 2}$$

$$= 2 - \frac{(2.1)^3 - 2*(2.1) - 5}{3*(2.1) - 2}$$

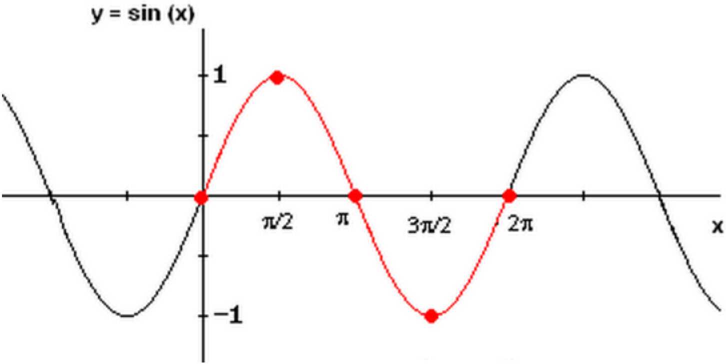
$$= 2.0946$$

Thus, the third approximation of the root $x_3 = 2.0946$.

Newton's Method: Example 2

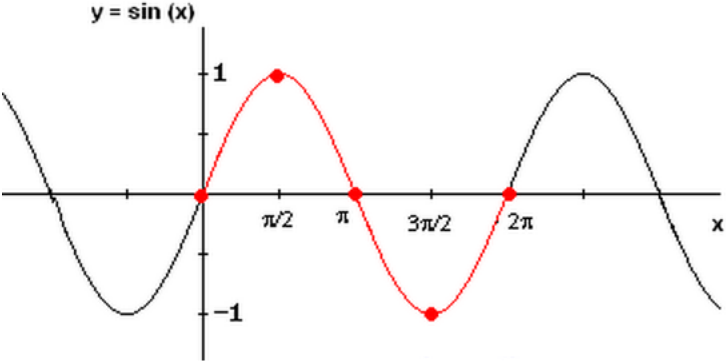
In some cases, Newton's Method fails for one reason or another.

Consider the function $f(x) = \sin(x)$.



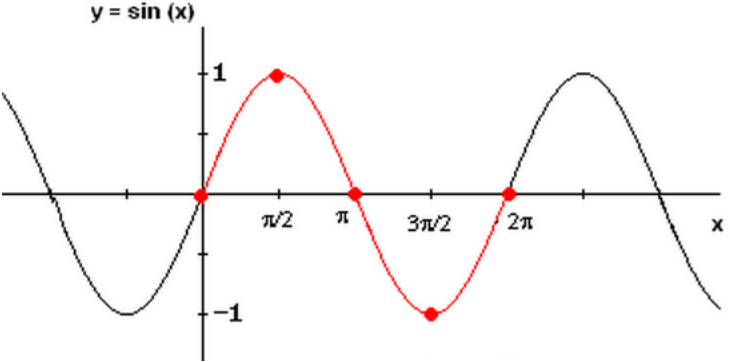
Newton's Method: Example 2

What happens if we take an initial guess $x_0 = \frac{\pi}{2}$?



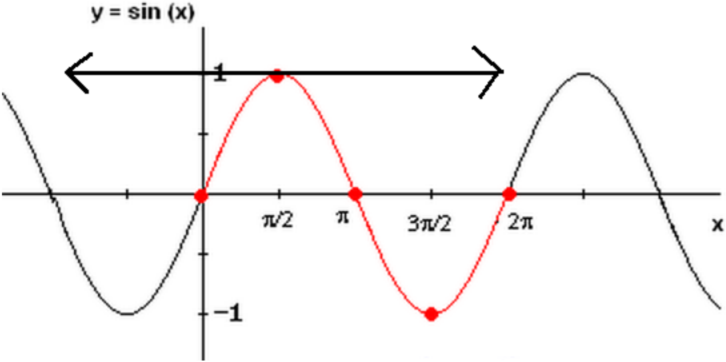
Newton's Method: Example 2

Why will Newton's Method fail to find a better x_1?



Newton's Method: Example 2

The tangent line to $f(x) = \sin(x)$ at $x = \frac{\pi}{2}$ is horizontal.



The tangent will not cross the x-axis.
There is no better estimate because there is no root.

Hints for Lab 8

1. The first graded problem students work is on page 3, entitled **1. A Graphical Example**
2. In working **2. Computational Example**, students should find the equation of each tangent line then set it equal to zero to find the next approximation.
3. In working **3. More Practice**, Students can use the short-cut formula

Ending Notes

Lab 8 is due Tuesday, 4/07/2015 in class.

- Proper notation is required.
- Work must be shown to receive credit.
- Tutoring available Monday night 5pm-8pm in Jones 112.

These slides are available at www.meganrosebryant.com.