# Spring 2015, Math 111
# Lab 9: The Definite Integral as the Area under a Curve

Megan Bryant

William and Mary

April 14, 2015

# Learning Objectives

Today, we will be looking at applications of Newton's Method.

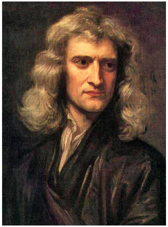
Learning Objectives:

- Introduce the geometric interpretation of the definite integral.
- Use simple geometry to define the definite integral as the difference of the antiderivative evaluated at the endpoints.
- Approximate the value of a definite integral (where simple geometry is not possible) using the sum of the areas of rectangles.

# Definite Integral: Historical Outline

Integrals were first studied using a method of exhaustion by Eudoxus in Ancient Greece (370 BC).

Little was achieved in the area of integration until the 17th century when the Fundamental Theory of Calculus was discovered by both Newton and Leibniz.



This theorem demonstrates a connection between integration and differentiation. It is this connection that we can exploit to calculate integrals.

# Definite Integral: Outline

So what is an integral?

The simplest notion is

$$\int f(x)dx$$

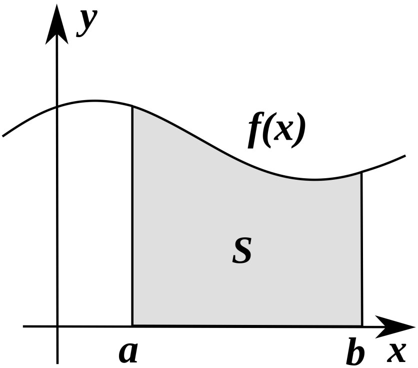
This the form of an indefinite integral. A definite integral is defined on a certain region

$$\int_a^b f(x)dx$$

We call this the definite integral of $f(x)$ from $a$ to $b$.

# Definite Integral: Intuition

Classically, the definite integral is defined to be 'the area under the curve $f(x)$ over the interval $[a, b]$'.



# Definite Integral: Example 1

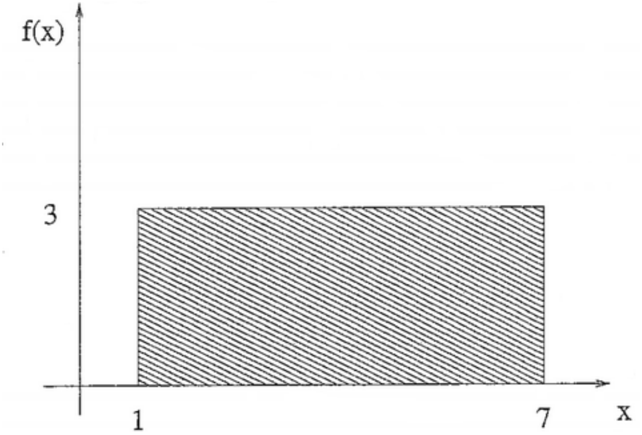
We can solve some integrals using simple geometry.

Let's consider the case when $f(x) = 3$ on the interval $[1, 7]$. We want to calculate the area under the curve.

Thus, we have the integral $\int_1^7 f(x)dx$.

# Definite Integral: Example 1

Graphically, we have the following.



# Definite Integral: Example 1

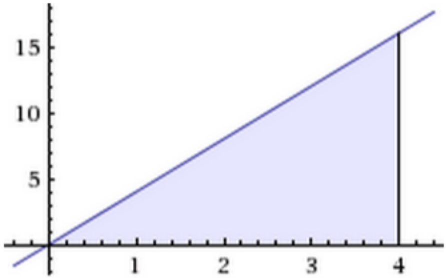
Now, we don't know how to calculate this integral yet. We do know how to find the area of a rectangle though. Thus we can say the following.

$$\int_{1}^{7} f(x)dx = (7-1)*(3-0)$$

$$\int_{1}^{7} 3dx = 18$$

# Definite Integral: Example 2

Consider the function $f(x) = 4x$ on the interval $[0, 4]$. Graphically we have the following.



# Definite Integral: Example 2

As before, we can use simple geometry to evaluate this integral.

$$\int_0^4 f(x)dx = \frac{1}{2} b * h$$

$$\int_0^4 4x dx = \frac{1}{2}(4-0)(4*4)$$

$$= \frac{4^3}{2}$$

$$= 32$$

# Definite Integral: Riemann Sums

Thus far, we know how to calculate basic geometric intervals. What about actual curves?

We can approximate the area under the curve by computing the sum of areas of many rectangles. The resulting estimate is called a Riemann sum after Bernhard Riemann (1826-1866).

# Definite Integral: Riemann Sums

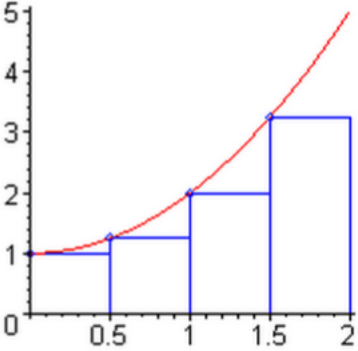
These approximations can be calculated as follows.

1. Split the interval into equal sub intervals.
2. In each sub interval, draw a rectangle whose height is given by the left or right endpoint.
3. Compute the areas of these rectangles and add them together.

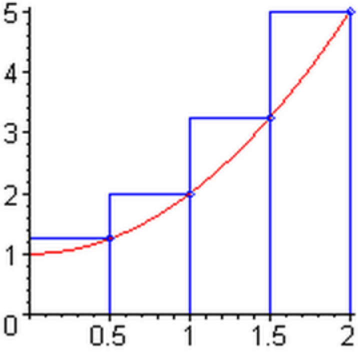
The result is an approximation of the area under the curve (the integral).

# Definite Integral: Riemann Sums

**Left**




**Right**



# Definite Integral: Riemann Sums

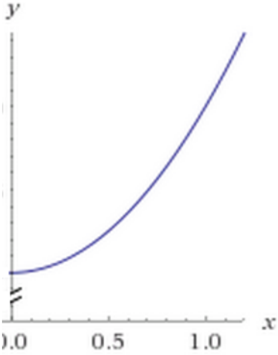
As we can see the left sum over estimates the area under the curve whereas the right sum underestimates the sum.

We can find a better estimate for the area under the curve by finding both the left and right sums and taking the average.

$$\text{Estimate: } \frac{L+R}{2}$$

# Definite Integral: Example 3

Consider the function $f(x) = x^2 + 1$. Graphically, we can visualize this as follows.



# Definite Integral: Example 3

Let's compute the **left** sum in the interval $[0, 2]$ with the width of each rectangle being $\frac{1}{2}$.

$$
\begin{aligned}
Area_L &= f(0) * \left(\frac{1}{2}\right) + f\left(\frac{1}{2}\right) * \left(\frac{1}{2}\right) + f(1) * \left(\frac{1}{2}\right) + f\left(\frac{3}{2}\right) * \left(\frac{1}{2}\right) \\
&= 1\left(\frac{1}{2}\right) + \frac{5}{4}\left(\frac{1}{2}\right) + 2\left(\frac{1}{2}\right) + \frac{13}{4}\left(\frac{1}{2}\right) \\
&= 3.75
\end{aligned}
$$

# Definite Integral: Example 3

Now, let's compute the **right** sum.

$$
\begin{aligned}
Area_L &= f\left(\frac{1}{2}\right) * \left(\frac{1}{2}\right) + f(1) * \left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right) * \left(\frac{1}{2}\right) + f(2) * \left(\frac{1}{2}\right) \\
&= \frac{5}{4}\left(\frac{1}{2}\right) + 2\left(\frac{1}{2}\right) + \frac{13}{4}\left(\frac{1}{2}\right) + 5\left(\frac{1}{2}\right) \\
&= 5.75
\end{aligned}
$$

# Definite Integral: Example 3

Now, we can find the average of the left and right sum to obtain the best estimate.

$$\text{Estimate:} \quad = \frac{L+R}{2} = \frac{3.75+5.75}{2} = 4.75$$

Note that the actual integral is approximately 4.67. Thus, the left and right sums give a good estimate of the integral.

# Hints for Lab 9

For problem #6, please use rectangle widths of $250K$ and then calculate both the left and right sums ($L$ and $R$). As with the Riemann Sums, the final estimate is the average of these two.

# Ending Notes

Lab 9 is due Tuesday, 4/21/2015 in class.

- Proper notation is required.
- Work must be shown to receive credit.
- Tutoring available Monday night 5pm-8pm in Jones 112.

These slides are available at www.meganrosebryant.com.