DISCRETE OPTIMIZATION

Megan Bryant
mrbryant@email.wm.edu

Department of Mathematics, The College of William and Mary, P.O. Box 8795, Williamsburg, VA 23187

Executive Summary

Homework 1.

Problems taken from *Optimization in Operations Research* by Ronald L. Rardin. Some definitions are quoted directly.

12-3

Consider the ILP

$$\begin{aligned} \max\ & 14x_1 + 2x_2 - 11x_3 + 17x_4 \\ \text{s.t. } & 2x_1 + x_2 + 4x_3 + 5x_4 \leq 12 \\ & x_1 - 3x_2 - 3x_3 - 3x_4 \leq 0 \\ & x_1 \geq 0 \\ & x_2, x_3, x_4 \in \{0, 1\} \end{aligned}$$

b.) *Determine whether the following is a constraint relaxation.*

$$\begin{aligned} \max\ & 14x_1 + 2x_2 - 11x_3 + 17x_4 \\ \text{s.t. } & 2x_1 + x_2 + 4x_3 + 5x_4 \leq 12 \\ & x_1 - 3x_2 - 3x_3 - 3x_4 \leq 0 \\ & x_1 \geq 0 \text{ and integer} \\ & x_2, x_3, x_4 \in \{0, 1\} \end{aligned}$$

By definition, model $(\widetilde{P})$ is a constraint relaxation of model (P) if every feasible solution to (P) is also feasible in $\widetilde{P})$ and both have the same objective function. Therefore, we know that the above ILP does not represent a constraint relaxation as an integrality constraint has been added to the variable x_1 thus making the search space more restricted.

d.) *Determine whether the following is a constraint relaxation.*

$$\begin{aligned} \max\ & 14x_1 + 2x_2 - 11x_3 + 17x_4 \\ \text{s.t. } & x_1 - 3x_2 - 3x_3 - 3x_4 \leq 0 \\ & x_1 \geq 0 \\ & x_2, x_3, x_4 = 0 \text{ or} \end{aligned}$$

If we refer again to the definition given in part **b.)**, we see that the above ILP is in fact a constraint relaxation as the removal of the first constraints results in a relaxed model $(\widetilde{P})$ in which every feasible solution to (P) is also feasible in $(\widetilde{P})$ and the objective function has remained the same.

12-4

Form the linear programming relaxation of each of the following ILPs.

b.) *Original ILP:*

$$\begin{aligned} \max\ & 9x_1 + 3x_2 + 4x_3 + 8x_4 \\ \text{s.t. } & 2x_1 + 2x_2 + 2x_3 + 3x_4 \leq 20 \\ & 29x_1 + 14x_2 + 78x_3 + 20x_4 \leq 200 \\ & x_1, x_2, x_3 = 0 \text{ or } 1 \\ & x_4 \geq 0 \end{aligned}$$

We know that linear programming relaxations of an integer linear program are formed by treating any discrete variables as continuous while retaining all other constraints. Therefore, we form the linear relaxation as follows,

$$\begin{aligned} \max\ & 9x_1 + 3x_2 + 4x_3 + 8x_4 \\ \text{s.t.}\ & 2x_1 + 2x_2 + 2x_3 + 3x_4 \leq 20 \\ & 29x_1 + 14x_2 + 78x_3 + 20x_4 \leq 200 \\ & 0 \leq x_1, x_2, x_3 \leq 1 \\ & x_4 \geq 0 \end{aligned}$$

12-11

The ILP

$$\begin{aligned} \min\ & 40x_3 + 500x_4 + 800x_5 + 900x_6 \\ \text{s.t.}\ & 10x_1 + 6x_2 + 2x_3 = 45 \\ & 2x_1 + 3x_2 + x_3 \geq 12 \\ & 0 \leq x_1 \leq 5x_4 \\ & 0 \leq x_2 \leq 5x_5 \\ & 0 \leq x_3 \leq 5x_6 \\ & x_4, x_5, x_6 = 0 \text{ or } 1 \end{aligned}$$

has the folllowing LP relaxation optimum,

$$\widetilde{\mathbf{x}} = (3.500, 1.667, 0, 0.70, 0.333, 0).$$

a.) *Determine the best bound on the ILP optimal solution value from relaxation results.*

We know that the optimal value of a linear programming relaxation to an integer linear program of a minimize model yields a lower bound to the original ILP.

Therefore we know that that the best lower bound on the ILP is 616.4.

b.) *Determine whether the relaxation optimum solves the full ILP. If not, round to an ILP feasible solution either by moving all binary values in the relaxation up to 1 or by moving all down to 0.*

We see that the optimal solution to the linear relaxation, $\widetilde{\mathbf{x}}$, violates several of the integrality constraints of the original ILP. Namely, the values for the variables $x_4, \& x_6$ are non integer in the linear relaxation solution. Therefore, the relaxation optimum does not solve the ILP.

We will thus round to an ILP feasible solution by rounding all binary values in the relaxation up to 1. This gives us the feasible solution of

$$\hat{\mathbf{x}} = (3.500, 1.667, 0, 1, 1, 0)$$

with a corresponding objective of $1,300$.

c.) *Combine parts* ***a.)*** *and* ***b.)*** *to determine the best upper and lower bounds on the ILP optimal solution value available from a combination of relaxation and rounding.*

We can conclude that the feasible objective solution obtained in part **b.**) constitutes an upper bound on the ILP optimal. Therefore, we may combine our results to arrive at the following bounds on the optimal ILP value

$$616.4 \leq \mathbf{x}^* \leq 1300.$$

12-15

The ILP

$$\begin{aligned} \min\ & 14x_1 + 16x_2 + 15x_3 \\ \text{s.t.}\ & x_1 + x_2 \geq 1 \\ & x_2 + x_3 \geq 1 \\ & x_1 + x_3 \geq 1 \\ & x_1, x_2, x_3 = 0 \text{ or } 1 \end{aligned}$$

has the following LP relaxation optimum $\widetilde{\mathbf{x}} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. *Determine whether each of the following is a valid inequality for the ILP, and if so, whether it would strengthen the original LP relaxation to add the inequality as a constraint.*

a.)

$$10x_1 + 10x_2 + 10x_3 \geq 25$$

We know that a linear inequality is a valid inequality for a given discrete optimization model if it holds for all (integer) feasible solutions to the model.

We consider below all possible solutions to the original ILP. Quick heuristic calculations allow us to determine which solutions are feasible and these are denoted accordingly.

(0,0,0)	I	(1,1,0)	F
(1,0,0)	I	(1,0,1)	F
(0,1,0)	I	(0,1,1)	F
(0,0,1)	I	(1,1,1)	F

We see that while solutions in which two of the three variables satisfy the constraints of the ILP, the do not satisfy the linear equality. Thus the linear equality in invalid for the ILP.

b.)

$$x_1 + x_2 + x_3 \geq 1$$

The linear inequality is valid for ILP as every solution that is feasible it is also feasible to the ILP.

We know that a relaxation is strong if its optimal value closely bounds that of the true model, and its optimal solution closely approximates an optimum in the full model. Therefore, to strengthen a relaxation, a valid inequality must cut off some feasible solutions to the current LP relaxation that are not feasible in the full ILP model.

Since the inequality does not exclude the previously obtained optimal solution to the linear relaxation, we know that adding the linear inequality to the original LP relaxation will not strengthen it.

c.)

$$x_1 + x_2 + x_3 \geq 2$$

We again see that the inequality is valid as all solutions that are feasible to the ILP are feasible to the inequality. Now we must consider whether or not it strengthens the LP relaxation.

We see that the optimal LP solution that was originally obtained, $\tilde{\mathbf{x}}$, would be excluded from the search space if the inequality was added as a constraint to the LP relaxation as it violates this additional constraint:

$$\frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1.5 \ngeq 2.$$

Therefore, we can conclude that this inequality is both valid and would strengthen the LP relaxation.

d.)

$$14x_1 + 20x_2 + 16x_3 \geq 28$$

On the surface, this inequality appears to be more complicated to evaluate. However, the criteria for validity is the same and the search space quite small. We can quickly see that every solution that is feasible to the ILP is also feasible to the inequality and it is therefore valid.

We must once again consider whether or not the inequality strengthens the LP relaxation. We see that the optimal LP solution that was originally obtained, $\tilde{\mathbf{x}}$, would be excluded from the search space if the inequality was added as a constraint to the LP relaxation as it violates this additional constraint:

$$14 * \frac{1}{2} + 20 * \frac{1}{2} + 16 * \frac{1}{2} = 25 \ngeq 28.$$

Therefore, we can conclude that this inequality is both valid and would strengthen the LP relaxation.

12-17

The fixed-charge ILP

$$\begin{aligned} \min\ & 6x_1 + 78x_2 + 200y_1 + 400y_2 \\ \text{s.t. }\ & 12x_1 + 20x_2 \geq 64 \\ & 15x_1 + 10x_2 \leq 60 \\ & x_1 + x_2 \leq 10 \\ & 0 \leq x_1 \leq 100y_1 \\ & 0 \leq x_2 \leq 100y_2 \\ & y_1, y_2 = 0 \text{ or } 1 \end{aligned}$$

has LP relaxation optimum

$$\widetilde{\mathbf{x}} = (0, 3.2), \widetilde{\mathbf{y}} = (0, 0.032)$$

.

a.) *Compute the smallest replacements for big-M values of 100 in this formulation that can be inferred simply by examining constraints of the model.*

Let us begin by clearly defining M_1 as the big-M value corresponding to y_1 and M_2 as the big-M value corresponding to y_2.

It is simple to determine the limiting values for the big-M's by critically examining the relevant constraints. In this case, we will be concerned with the two '$\leq'$ constraints and the ratio of the variable coefficients to the RHS values.

Thus, we see that for x_1,

$$M_1 = \min\{\frac{60}{15}, \frac{10}{1}\} = \min\{4, 10\} = 4$$

.

Similarly, for x_2,

$$M_2 = \min\{\frac{60}{10}, \frac{10}{1}\} = \min\{6, 10\} = 6$$

.

b.) *Show that the LP relaxation optimum will change if the lower big-M's of part* **a**.) *are employed.*

The original LP relaxation has an LP optimum

$$\widetilde{\mathbf{x}} = (0, 3.2), \widetilde{\mathbf{y}} = (0, 0.032).$$

With our new big-M values, we have the following updated LP relaxation inequalities:

$$0 \leq x_1 \leq 4y_1$$
$$0 \leq x_2 \leq 6y_2$$

However, when we evaluate the inequalities using the LP relaxation optimum, we see that the second constraint is violated as follows

$$0 \leq 3.2 \not\leq 0.192.$$

Therefore, the LP relaxation optimum must change because it is no longer feasible.