MthS 4120/6120, Fall 2013

HW #1. Due Thursday, August 29th.

- Read Francis Su's essay on good mathematical writing (posted on the course website) and answer the following questions.
  (1) What is a good rule of thumb for what you should assume of your audience as you write your homework sets?
  (2) Is chalkboard writing formal or informal writing?
  (3) Why is the proof by contradiction on page 3 not really a proof by contradiction?
  (4) Name three things a lazy writer would do that a good writer would not.
  (5) What's the difference in meaning between these three phrases?

  "Let $A = 12$."   "So $A = 12$."   "$A = 12$."
- Read the article *Group Think* by Steven Strogatz, which appeared in the New York Times in 2010. (It's posted on the course webpage.)
- Read VGT, Chapters 1 & 2.
- VGT Exercises 1.2, 1.8–1.13.

HW #2. Due Thursday, September 5th.

- Read VGT, Chapter 3.
- VGT Exercises 1.14, 2.2, 2.6, 2.9, 2.11, 2.12, 2.17, 2.19.
- Draw a Cayley diagram for the "square puzzle group" using 2 generators that are both reflections. Compare this to the Cayley graph using a rotation and a reflection as the generating set (which we did in class; see Exercise 2.8).
- Do Exercise 2.18. Compare your Cayley diagram for this problem to the one in Exercise 2.5, which we did in class. What can you say about these two groups?

HW #3. Due Tuesday, September 10th.

- Read VGT, Chapter 3.
- VGT Exercises 3.8, 3.11. Check the online VGT errata for a correction to 3.11.
- Recall that the identity of a group G is an element $e \in G$ satisfying $ge = eg = g$ for all $g \in G$. Prove that in any group, the identity element is unique. [*Hint*: Start by assuming that e and e' are both identity elements. Can you prove that $e = e'$?]
- There is a subtle "problem" with the contra dancing example from VGT, which is why we did not spend much time on it in class. In this exercise, I want you to discover what this problem is. As a hint, what is the result of doing "Circle right" + "Ladies chain"? Try starting from several different initial configurations of the dancers. What problem do you see? Can you propose a way to fix it? Justify your argument. You may want to refer to the "square puzzle" and its Cayley graph.

HW #4. Due Tuesday, September 17th.

- Read VGT, Chapter 4.
- VGT Exercises 4.1, 4.2, 4.13, 4.16, 4.19–26. For any problem involving the *Quaternion group* Q_4, please use the Cayley diagram shown in the online VGT errata instead of the one on page 53 in the textbook.

HW #5a. Due Tuesday, September 24th.

- Read: VGT, Chapter 5, and Exercises 5.1, 5.11, 5.12, 5.21, 5.22.
- Do: VGT Exercises 5.3 (give a *brief* justification for each true/false), 5.10(a,b,c,d), 5.13, 5.14, 5.15(a,e,f), 5.30, 5.33.

HW #5b. Due Tuesday, October 1st.

- Read: VGT, Chapter 5, and Exercises 5.25–27, 5.32, 5.36.
- Do: VGT Exercises 5.20, 5.34, 5.35, 5.37, 5.41(b), 5.42.
- Compute the product of the following permutations. Your answer for each should be a single permutation written in cycle notation as a product of disjoint cycles.
  a. $(1\;3\;2)\,(1\;2\;5\;4)(1\;5\;3)$ in S_5
  b. $(1\;5)\,(1\;2\;4\;6)\,(1\;5\;4\;2\;6\;3)$ in S_6.
- Write out all $4! = 24$ permutations in S_4 in cycle notation. Additionally, write each as a product of transpositions, and decide if they are even or odd. Which of these permutations are also in A_4?
- Exercise 4.6(c) gives a Cayley diagram for A_4, but the elements are named with letters instead of permutations:
  $$A_4 = \{e, a, a^2, b, b^2, c, c^2, d, d^2, x, y, z\}.$$
  Redraw this Cayley diagram but label the nodes with the 12 even permutations from the previous problem. That is, you need to determine which permutation corresponds to a, which to b, and so on.

HW #6a. Due Tuesday, October 8th.

- Read: VGT, Chapter 6, and exercises 6.15, 6.16.
- Do: VGT Exercises 6.5, 6.7–9, 6.11–13, 6.20, 6.28.
- Prove that every subgroup of a cyclic group is cyclic. (Do not assume that G is finite).

HW #6b. Due Thursday, October 17th.

- Read: VGT, Chapter 6, and exercises 6.15, 6.16.
- Do: VGT Exercises 6.17, 6.18, 6.29.
- Do: VGT Exercise 6.22–24. Additionally, construct the subgroup lattice, or Hasse diagram, for C_{24}. In all your Hasse diagrams, label each edge with the corresponding *index*.
- Prove the following (do not refer to Cayley diagrams):
  (a) If $\mathcal{H}$ is a collection of subgroups of G, then the intersection $\bigcap_{H\in\mathcal{H}} H$ is a subgroup of G.
  (b) If $S \subset G$, then $\langle S\rangle$ is the intersection of all subgroups containing S. [*Hint*: One way to prove that $A = B$ is to show that $A \subset B$ and $B \subset A$.]
- (a) Prove that if $x \in H$, then $xH = H$. What is the interpretation of this statement in terms of the Cayley diagram?
  (b) Prove, that if $b \in aH$, then $aH = bH$. (Use the definition of a coset: $aH = \{ah \colon h \in H\}$.)

HW #7a. Due Wednesday, October 23rd.

- Read: VGT, Chapter 7.1–3.
- VGT Exercises (Products): 7.7, 7.8, 7.9(use the "correct" Cayley diagram of Q_4 from the online errata), 7.12(do the "algebraic" proof), 7.13.
- VGT Exercises (Quotients): 7.17, 7.18(c,d,e,f,g,h), 7.24.
- Recall that G/H is the set of (left) cosets of H in G. We defined a binary operation on G/H of left cosets by $aH \cdot bH = abH$. In this exercise, you will see further motivation for this definition. Given $a, b \in G$, define the sets
  $$aHbH = \{ah_1bh_2 \colon h_1, h_2 \in H\} \qquad \text{and} \qquad abH = \{abh \colon h \in H\}\,.$$
  Prove that if $H \lhd G$, then $aHbH = abH$ (show that an arbitrary element of abH is in $aHbH$, and vice-versa). Comment on how this relates to quotient groups.

HW #7b. Due Monday, October 28th.

- Read VGT, Chapter 7.3–4.
- Prove or disprove the following statements (without refering to Cayley diagrams). To disprove something, all you need to do is find a single example where it fails.
  (a) Every subgroup of an abelian group is normal.
  (b) Every quotient of an abelian group is abelian.
  (c) If $K \lhd H \lhd G$, then $K \lhd G$.
  (d) If $K \leq H \leq G$ and $K \lhd G$, then $K \lhd H$.
- Recall that the *center* of a group G is the set

$$\begin{aligned} Z(G) &= \{z \in G \mid gz = zg,\ \forall g \in G\} \\ &= \{z \in G \mid gzg^{-1} = z,\ \forall g \in G\}. \end{aligned}$$

  (i) Prove that $Z(G)$ is a subgroup of G. (That is, show that it contains the identity, inverses, and is closed under the group operation.)
  (ii) Prove that $Z(G)$ is a *normal* subgroup of G. (That is, show that for any $x \in Z(G)$, the element $gxg^{-1} \in Z(G)$.)
- VGT Exercises (Normalizers): 7.25(c,d), 7.26(c,d), 7.27.
- VGT Exercises (Conjugacy): 7.29, 7.32, 7.33(a,b).

HW #8a. Due Friday, November 1st.

- Read VGT, Chapter 8.1–8.2.
- VGT Exercises (Embeddings & quotient maps): 8.2–5, 8.10, 8.12, 8.15–17, 8.36. (For 8.2 give a *brief* justification for each true/false.)

HW #8b. Due Tuesday, November 5th.

- Read VGT, Chapter 8.3.
- VGT Exercises (FHT): 8.13, 8.14, 8.40(book has a typo, see online errata).
- VGT Exercises (Modular arithmetic): 8.20, 8.22, 8.23.
- (a) Prove that if $H < G$, then $H \cong gHg^{-1}$, for any $g \in G$. (Recall that we showed in class that gHg^{-1} is always a subgroup of G.)
  (b) Use Part (a) to show that in any group, $|xy| = |yx|$.
- Prove that if A and B are normal subgroups of G, and $AB = G$, then

$$G/(A \cap B) \cong (G/A) \times (G/B).$$

  [*Hint*: Construct a homomorphism $\phi\colon AB \to (G/A) \times (G/B)$ that has kernel $A \cap B$, then apply the FHT.]

HW #8c. Due Friday, November 8th.

- Read VGT, Chapter 8.4–8.5.
- VGT Exercises (Finite abelian groups): 8.50.
- VGT Exercises (Misc.): 8.39(a), 8.41–43.
- For each order, list all abelian groups of that order (up to isomorphism), as a product of cyclic groups of prime-power order.
  (a) $32 = 2^5$
  (b) $60 = 2^2 \cdot 3 \cdot 5$
  (c) $108 = 2^2 \cdot 3^3$

  An alternative way to write a finite abelian group is

$$A \cong C_{k_1} \times C_{k_2} \times \cdots \times C_{k_\ell}\,,$$

  where $k_1 \mid k_2 \mid \cdots k_{\ell-1} \mid k_\ell$ (but the k_i's need not be prime powers). For each order in the previous question, list all abelian groups of that order in this manner.
- Let A and B be normal subgroups of G.
  (a) Prove that the set $AB := \{ab : a \in A,\ b \in B\}$ is a subgroup of G.

(b) Prove that $B \lhd AB$ and $A \cap B \lhd A$.
(c) Prove that $A/(A \cap B) \cong AB/B$. [*Hint:* Construct a homomorphism $\phi\colon A \to AB/B$ that has kernel $A \cap B$, then apply the FHT.]
(d) Draw a diagram, or lattice, of G and its subgroups AB, A, B, and $A \cap B$. Interpret the result in Part (c) in terms of this diagram.

HW #9a. Due Thursday, November 14th.

- Read VGT, Chapters 9.1, 9.2, 9.3. Read: Exercises 9.3, 9.12.
- Do: VGT Exercises 9.4, 9.7, 9.9, 9.10.
- Repeat the exercise from the class lecture notes for several other groups: Let S be the set of "binary squares". Draw an action diagram for each of the following group actions:
  (1) $\phi\colon V_4 \longrightarrow \text{Perm}(S)$, where $\phi(h)$ reflects each square horizontally, and $\phi(v)$ reflects each square vertically;
  (2) $\phi\colon C_4 \longrightarrow \text{Perm}(S)$, where $\phi(1)$ rotates each square 90° clockwise;
  (3) $\phi\colon D_4 \longrightarrow \text{Perm}(S)$, where $\phi(r)$ rotates each square 90° clockwise, and $\phi(f)$ reflects each square about a vertical axis.
  Additionally, pick an element in each orbit and find its stabilizer.
- Let G be a group of order 15, which acts on a set S with 7 elements. Show that the group action has a fixed point.
- Let G act on itself (i.e., $S = G$) by conjugation.
  (1) Show that the set of fixed points of this action is $Z(G)$, the center of G.
  (2) Prove that if G is a p-group (i.e., $|G| = p^n$ for some prime p), then $Z(G)$ is nontrivial.
  (3) A group is *simple* if its only normal subgroups are G and $\{e\}$. Use Part (b) to completely classify all simple p-groups.

HW #9b. Due Thursday, November 21st.

- Read VGT, Chapters 9.4, 9.5. Read: Exercise 9.17.
- Do: VGT Exercises 9.21, 9.22, 9.23
- Prove that a Sylow p-subgroup of G is normal if and only if it is the unique Sylow p-subgroup of G. [*Hint*: Recall that gHg^{-1} is always a subgroup of G.]
- Recall that a group G is called *simple* if its only normal subgroups are G and $\{e\}$.
  (a) Show that there is no simple group of order pq, where $p < q$ and are both prime. [*Hint*: Show that G contains a unique Sylow subgroup for some prime.]
  (b) Show that there is no simple group of order $108 = 2^2 \cdot 3^3$.
- Suppose that $H \lneq G$, and let $S = G/H$. Let G act on S, where $\phi(g)\colon xH \mapsto gxH$.
  (a) Show that if $|G|$ does not divide $[G : H]!$, then G cannot be simple.
  (b) Use (a), together with the Sylow theorems, to show that any group of order 36 cannot be simple.

HW #10a. Due Tuesday, November 26th.

- Read VGT, Chapters 10.1–10.4.
- VGT Exercises 10.8, 10.15, 10.22, 10.29(book has a typo, see online errata), 10.30.

HW #10b. Due Tuesday, December 3rd.

- Read VGT, Chapters 10.5, 10.6.
- VGT Exercises 10.13, 10.14, 10.16,
- For each polynomial, use Eisenstein's criterion to test for irreducibility. For the ones for which it fails, try to factor into irreducible factors.
  (a) $x^4 - 3x^3 + 12x^2 + 51$
  (b) $60x^2 + 50x - 10$
  (c) $x^3 - 6x^2 + 10x + 2$
  (d) $x^4 + 7x^2 + 10$

HW #10c. Due Thursday, December 12th.

- Read VGT, Chapters 10.7.
- VGT Exercises 10.3, 10.18–20, 10.26.
- Let $f(x) = x^4 - 7$.
  - (i) Determine the splitting field F of $f(x)$. Give an explicit basis. What does this tell you about the order of the Galois group $G = \mathrm{Gal}(f)$?
  - (ii) Compute the Galois group of f. Write down the two automorphisms that generate it (call them r and f).
  - (iii) Draw the subgroup lattice of G. Label the edges by index, and circle the subgroups that are normal in G.
  - (iv) Draw the subfield lattice of F. Label the edges by degree, and circle the subfields that are normal extensions of $\mathbb{Q}$.
  - (v) For each intermediate subfield K, write down the largest subgroup of G that fixes K.
  - (vi) For each subgroup $H < G$, write down the largest intermediate subfield fixed by H.
- The roots of the polynomial $f(x) = x^n - 1$ are the n complex numbers $\{e^{2k\pi i/n} : k = 0, 1, \ldots, n-1\}$. These are called the n^{th} *roots of unity*. A *primitive* root of unity is $\zeta = e^{2k\pi i/n}$ for which $\gcd(n,k) = 1$. Note that the roots of unity form a group under multiplication (see Chapter 5, slide 6/42).
  - (i) For $n = 3, \ldots, 8$, sketch the roots of $x^n - 1$ in the complex plane. Use a different set of axes for each n.
  - (ii) For each $n = 3, \ldots, 8$, write $x^n - 1$ as a product of irreducible factors. [*Hint*: Try Googling *cyclotomic polynomial*.]
  - (iii) The Galois group of $x^n - 1$ is the group U_n, or $(\mathbb{Z}/n\mathbb{Z})^\times$ (see Exercise 8.41). Justify this by explicitly describing the automorphisms of the splitting field that generate the Galois group.