NETWORK OPTIMIZATION

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Executive Summary

Homework 1.
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1.1 *An undirected graph $G = (N, A)$ is said to be bipartite if it is possible to partition the nodes into sets L, R such that each arc (i, j) has one endpoint in L and one endpoint in R.*

Prove the following statement:

Theorem 1. *An undirected graph G is bipartite if and only if every closed walk in G has an even number of arcs.*

$\rightarrow$ *If an undirected graph $G = (N, A)$ is bipartite, then every closed walk in G has an even number of arcs.*

Proof. Let us assume that there exists some closed walk, w, in G such that w has an odd number of arcs.

We know that by definition w is a sequence of nodes, n_i, in N and arcs, a_i, in A,

$$n_1, a_1, n_2, a_2, \ldots, n_{2k+1}, a_{2k+1}, n_{2k+2}$$

in which the endpoints of arc a_i are the surrounding nodes n_i and n_{i+1}. In order for the walk to be a closed walk, we know that the starting node must also be the ending node, i.e. $n_1 = n_{2k+2}$.

Since we assumed that the number of arcs in closed walk w is odd, the index $2k + 1$ is used to denote the general form of an odd integer where $k \in \mathbb{N}$. This implies that the terminal node with index $2k + 2 = 2(k + 1)$ is even by definition.

By the definition of bipartite, we know that if one endpointof arc a_i, node n_i, is in set L, then the corresponding endpoint of a_i, node n_{i+1}, must be in set R.

Let us assume without loss of generality that $n_1 \in L$. This implies that the corresponding endpoint of arc a_1, node $n_2 \in R$.

By the definition of a walk, this endpoint, n_2, is also an endpoint of the succeeding arc a_2. However, since n_2 is in R and G is bipartite, the remaining endpoint of a_2, node n_3, must be in set L. Continuing on in this fashion, we deduce that all nodes with an even index must be in the set R and all nodes with an odd index must be in set L.

We have previously noted that if there are an odd number of arcs in walk w, then the terminal node n_{2k+2} must have an even index. Therefore, we know that node $n_{2k+2} \in R$.

However, as stated earlier, in order for the walk to be closed, $n_1 = n_{2k+2}$ which implies that $n_{2k+2} \in L$, which is a direct contradiction.

Thus, if G is an undirected and bipartite graph, then every closed walk in G must have an even number of arcs. $\square$

$\leftarrow$ *If every closed walk on the undirected graph $G = (N, A)$ has an even number of arcs, then G is bipartite.*

Proof. We may begin by assuming the graph G is connected (in cases in which G is not connected, we may consider each connected component seperately and proceed accordingly).

Let us examine the implications if G has either one node or no nodes at all. In either case, G is trivially bipartite and thus require no further consideration.

Therefore, let us assume without loss of generality that G is a connected graph with at least two nodes.

Let us choose a random node n_0 of G. Since G is connected, we know that there is a path from n_0 to every other node of G.

We will partition the remaining nodes into two sets, E and O. Let E contain all nodes which have a path of even length originating from n_0. Similarly, O will contain all nodes which have a path of odd length originating from n_0.

We know that $E \cup O = G$ since G is connected. Furthermore, we can see that the intersection of the two sets is empty.

Let us assume for the sake of contradiction that there exists some node n_i such that $n_i \in E \cap O$. This implies that there exists both paths even length and odd length from our random node n_0 to node n_i.

This implies that the length of the resulting closed walk is odd as it axiomatic that an even number plus and an odd number is an odd number. This result is a direct contradiction to the fact that every closed walk in G has a even number of arcs. Therefore, we know that the intersection of E and O must be empty and the sets partition the nodes of graph G.

Now let us assume that there exists an arc, a_i, in A such that both nodes, n_j and n_k are in set E. Then there exists an even length path from n_j to n_k and an even length path from n_k to n_j. Again, it is trivial to show that two even lengths plus a length of one results in an odd length. Therefore, these paths form a closed walk of odd length with arc a_i. This is a contradiction to our premise and thus a_i cannot connect two nodes in set E.

We may similarly show that there cannot exist an arc connecting two nodes in set O as it would once again result in a closed walk of odd length.

Since G is a connected graph and we have demonstrated that no arcs may connect two nodes in the same set, we may conclude that E and O partition the nodes into two sets such that each arc (i, j) has an endpoint in E and an endpoint in O and the graph is bipartite.

□

1.2 *Use induction to show that postage of six cents or more can be achieved by using only 2 cents and 7 cent stamps.*

Proof. We want to prove that the following proposition is true for all $n \geq 6$:

$P(n)$: Any denomination of postage can be achieved using only 2 cent and 7 cent stamps.

Basis Steps: $P(6) : 6 = 2 * 3 + 7 * 0 = 6$
$P(7) : 7 = 2 * 0 + 7 * 1 = 7$
$P(8) : 8 = 2 * 4 + 7 * 0 = 8$
$P(9) : 9 = 2 * 1 + 7 * 1 = 9$
$P(10) : 10 = 2 * 5 + 7 * 0 = 10$

Upon completion of these base cases, it become evident that there are two seperate algorithims at work. Therefore we will formulate two seperate cases. Together, these propositions will consider all cases of the original proposition, however we will prove them seperately.

$$\text{Case 1: } P(n)_{even} : k = 2 * \left(\frac{k}{2}\right) + 7 * 0$$

$$\text{Case 2: } P(n)_{odd} : k = 2 * \left(\frac{k-7}{2}\right) + 7 * 1$$

Inductive Hypothesis:
We will assume that $P(k)$ is true for all $6 \leq k$, where $k \in \mathbb{Z}$.

Inductive Step:
We must prove that $P(k + 1)$ is true.

Case 1, $k+1$ is even:

$$\begin{aligned} P(k+1) : k+1 &= 2 * \left(\frac{(k+1)}{2} \right) + 7 * 0 \\ &= 2 * \left(\frac{k}{2} + \frac{1}{2} \right) + 7 * 0 \\ &= 2 * \left(\frac{k}{2} \right) + 7 * 0 + 1 \\ &= P(k)_{even} + 1 \\ &= k+1 \end{aligned}$$

Case 2, $k+1$ is odd:

$$\begin{aligned} P(k+1) : k+1 &= 2 * \left(\frac{(k+1)-7}{2} \right) + 7 * 1 \\ &= 2 * \left(\frac{(k-7)+1)}{2} \right) + 7 * 1 \\ &= 2 * \left(\frac{k-7}{2} \right) + 7 * 1 + 1 \\ &= P(k)_{odd} + 1 \\ &= k+1 \end{aligned}$$

Therefore, we have proven our each case of our original proposition and thus proved that any denomination of postage can be achieved using only 2 and 7 cent stamps.

□

1.3 *Can you find what is wrong with the following inductive argument:*

Theorem 2. *All people have the same height.*

Proof: We use induction on the number of people. Base Case: 1 Person. Clearly if there is only one person, then all people have the same height. Inductive step: The inductive hypothesis is that for any group of n people, all have the same height. We show that his implies that for any group of $n+1$ people, all have the same height. Well, given any group of $n+1$ people, if we exclude the last person, we get a group of n people. By the inductive hypothesis, these n people all have the same height. We could also exclude the first person, and we would again get a group of n people, so also the last n people have the same height. Therefore, all $n+1$ people have the same height.

Counter: We see that a misstep is made when we assume that the group of people being considered is larger than 2. Given that $n = 2$, we encounter a significant problem when we exclude the last person.

Let A represent the set that is obtained after excluding the 'last person'. If $n = 2$, then the set A consists entirely of our initial person.

Similarily, let set B represent the set that is obtained by excluding the 'first person'. If $n = 2$, then set B consists entirely of our last person.

Our base case tells us that in sets containing only one person the proposition is satisfied and all people have the same height.

However, we know that $A \cap B = \emptyset$ since the two sets do not contain any common elements as people are distinct.

Therefore, we cannot assume that the induction hypothesis holds true for $n = 2$. Thus we cannot prove the hypothesis true for any cases larger than 2 and the proof is invalidated.