Advanced Calculus II
Unit 6.1
Problems 1b, 7, 9, 11, 12

Megan Bryant

September 3, 2013

6.1.1b

Given that $f(x) = 1 - x^2, x \in [-1, 2]$, $P = \{-1, -\frac{1}{2}, \frac{1}{2}, 1, \frac{3}{2}, 2\}$, *Find the upper and lower sums.*

$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i$ where $M_i = sup\{f(t) : x_{i-1} \leq t \leq x_i\}$.

$M_1 = .75$
$M_2 = 1$
$M_3 = .75$
$M_4 = 0$
$M_5 = -1.25$

$\Delta x_1 = .5$
$\Delta x_2 = 1$
$\Delta x_3 = .5$
$\Delta x_4 = 1$
$\Delta x_5 = .5$

Which implies $U(P, f) = .75 * .5 + 1 * 1 + .75 * .5 + 0 * 1.5 + -1.25 * .5 = 1.125$

$L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i$ where $m_i = inf\{f(t) : x_{i-1} \leq t \leq x_i\}$.

$m_1 = 0$
$m_2 = .75$
$m_3 = .0$
$m_4 = -1.25$
$m_5 = -3$

$\Delta x_1 = .5$
$\Delta x_2 = 1$
$\Delta x_3 = .5$
$\Delta x_4 = 1$
$\Delta x_5 = .5$

Which implies $L(P, f) = 0 * .5 + .75 * 1 + 0 * .5 + -1.25 * 1.5 + -3 * .5 = -1.375$

6.1.7

a.)Suppose f is continuous on [a,b] with $f(x) \geq 0$ for all $x \in [a, b]$. If $\int_a^b f = 0$, prove that $f(x) = 0$ for all $x \in [a, b]$.

Suppose there exists some $x_0 \in [a, b]$ such that $f(x_0) \geq 0$. Let $\epsilon = \frac{f(x_0)}{2}$. Since f is continuous, there exists $\delta \geq 0$ such that
$0 \leq |f(x_0) - f(x)| \leq \frac{f(x_0)}{2} = \epsilon$ for $x \in [a.b]$.

This implies $f(x_0) - f(x) \leq |f(x_0) - f(x)| < \frac{f(x_0)}{2} = \epsilon$.

Which implies $f(x) > \frac{f(x_0)}{2} > 0$.

Let P be a partition of $[a, b]$ where $P = \{a, x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}, b\}$

Thus, $L(P, f) \geq \delta * \epsilon = \delta * \frac{f(x_0)}{2}$.

Which implies, $supL(P, f) \geq L(P, f) \geq \frac{\delta * f(x_0)}{2} > 0$. But, this is a contradiction because we have $\int_a^b f = 0 = supL(P, f) = 0$ because f is continuous and thus Reimann integrable. Thus we have a contradiction and $f(x) = 0$ for all $x \in [a, b]$.

b.) A counterexample would be the following function,

$$f(x) = \begin{cases} 1 & x = \frac{1}{n}, \forall n \in N \\ 0 & otherwise \end{cases}$$

Thus, $f(x) \in R[a, b] and \int_a^b f(x)dx = 0$

6.1.9

Suppose f is a nonegative Reimann integrable function on $[a,b]$ satisfying $f(r)=0$ for all $r \in Q \bigcap [a,b]$. Prove that $\int_a^b f = 0$.

Let $P = \{x_0, x_1, \cdots, x_n\}$ such that $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$.

Since f is a nonnegative Reimann integrable function on $[a,b]$, we know that $\underline{\int_a^b} f(x)dx = \int_a^b f(x)dx = \overline{\int_a^b} f(x)dx$.

Since f is nonnegative, $f(x) \geq 0$ for all $x \in [a.b]$.

This implies for each $i = 1, 2, \cdots, n$, $m_i = inf\{f(t) : x_{i-1} \leq t \leq x_i\} = 0$

Thus $L(P,F) = \sum_{i=1}^{n} m_i \Delta x_i = 0$.

Since $\overline{\int_a^b} f(x)dx = sup\{L(P,f) : Pisapartitionof[a,b]\} = sup\{0\} = 0$, we can conclude that $\underline{\int_a^b} f(x)dx = \int_a^b f(x)dx = \overline{\int_a^b} f(x)dx = 0 = \int_a^b f$.

6.1.11

Suppose f is monotone increasing on $[a,b]$. For $n \in N$, set $h = (b-a)/n$. Let $P_n = \{x_0, x_1, ..., x_n\}$ where for each $k = 0, ..., n$, $x_k = a + kh$.

a.)*Prove that* $0 \leq U(P_n, f) - \int_a^b f \leq \frac{(b-a)}{n}[f(b) - f(a)]$.

Since every subinterval is of equal width and the partition is regular, $h = \frac{(b-a)}{n} = \Delta x$.

$x_k = a + kh$ implies that $x_0 = a$ and $x_n = b$

Since f is monotone increasing, we can infer that $x_0 = a \leq f(t) \leq x_n = b$ for all $t \in [a,b]$.

By theorem 6.1.8, we know that since f is monotone increasing on $[a,b]$, f is Reimann integrable on $[a,b]$.

This implies that that $\underline{\int_a^b} f(x)dx = \int_a^b f(x)dx = \overline{\int_a^b} f(x)dx$.

Since $\underline{\int_a^b} f(x)dx \leq U(P,f)$, $\overline{\int_a^b} f(x)dx \geq L(P,f)$, and $\underline{\int_a^b} f(x)dx \leq \overline{\int_a^b} f(x)dx$, we know that $0 \leq \underline{\int_a^b} f(x)dx - \overline{\int_a^b} f(x)dx \leq U(P,f) - L(P,f)$

Thus, $0 \leq U(P,f) - \underline{\int_a^b} f = U(P,f) - sup\{L(P,f)\} \leq \Delta x(f(b) - f(a))$

But,$U(P,f) - sup\{L(P,f)\} \leq U(P,f) - L(P,f) = \sum_{i=1}^{n}[f(x_i) - f_(x_{i-1})]\Delta x_i =$
$h = \sum_{i=1}^{n}[f(x_i - f(x_{i-1})] = \frac{(b-a)}{n}[f(b) - f(a)]$.

Thus $0 \leq U(P_n, F) - \int_a^b f \leq \frac{(b-a)}{n}[f(b) - f(a)]$.

b.) Prove that $\int_a^b f = \lim_{n\to+\infty} U(P,f)$. By theorem 6.1.8, we know that since f is monotone increasing on $[a,b]$, f is Reimann integrable on $[a,b]$.

This implies that that $\underline{\int_a^b} f(x)dx = \int_a^b f(x)dx = \overline{\int_a^b} f(x)dx$.

Since $L(P_n, f) \leq \underline{\int_a^b} f(x)dx$ and $\overline{\int_a^b} f(x)dx \leq U(P_n, f)$,
$0 \leq U(P_n, f) - \overline{\int_a^b} f(x)dx = U(P_n, f) - \underline{\int_a^b} f(x)dx \leq U(P_n, f) - L(P_n, f)$.

Since the limit of the right hand side is zero, we can conclude by the squeeze theorem that
$\lim_{n\to+\infty} U(P_n, f) = \overline{\int_a^b} f(x)dx = \int_a^b f$

6.1.12

a.)$\int_0^1 xdx =$

We know from the previous exercise that $\int_a^b f = \lim_{n\to+\infty} U(P,f)$.

Let $P_n = \{0, \frac{1}{N}, \frac{2}{N}, \cdots, 1\}$. Thus,

$U(P_n, f) = \sum_{i=1}^{N} \sup_{[\frac{i-1}{n}, \frac{i}{n}]} x \cdot \Delta x_i = \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{1}{n} = \frac{1}{n^2}\frac{n(n+1)}{2} = \frac{1}{2} + \frac{1}{2n}$

Thus $\lim_{n\to+\infty} U(P,f) = \lim_{n\to+\infty} \frac{1}{2} + \frac{1}{2n} = \frac{1}{2} = \int_0^1 xdx$.

b.)$\int_{-2}^1 (3x-2)dx =$

We know from the previous exercise that $\int_a^b f = \lim_{n\to+\infty} U(P,f)$.

$\Delta x = \frac{1-(-2)}{n} = \frac{3}{n}$

Let $P_n = \{-2 + \frac{3k}{n} : k = 0, 1, \cdots, n\}$.Thus,

$U(P_n, f) = \sum_{k=1}^{n} f(-2 + \frac{3k}{n})(\frac{3}{n}) = \sum_{k=1}^{n} \frac{3}{n}(-6 + \frac{9k}{n} - 2) =$
$\sum_{k=1}^{n} \frac{-24}{n} + \frac{27}{n^2} \sum_{k=1}^{n} k = -24 + 27(\frac{1}{2} + \frac{1}{n})$

Thus $\lim_{n \to +\infty} U(P, f) = \lim_{n \to +\infty} -24 + 27(\frac{1}{2} + \frac{1}{n}) = \frac{-21}{2} = \int_{-2}^{1} (3x - 2)dx$.

c.)$\int_0^1 x^3 dx =$

We know from the previous exercise that $\int_a^b f = \lim_{n \to +\infty} U(P, f)$.

$\Delta x = \frac{1-0}{n} = \frac{1}{n}$

Let $P_n = \{\frac{k}{n} : k = 0, 1, \cdots, n\}$.Thus,

$U(P_n, f) = \sum_{k=1}^{n} f(\frac{k}{n})(\frac{1}{n}) = \sum_{k=1}^{n} (\frac{k}{n})^4 * \frac{1}{n} = \frac{1}{n^5} \sum_{k=1}^{n} k^4 =$
$\frac{1}{4n^5} * (\frac{n}{2}(n+1))^4 = \frac{n^3}{64} + \frac{n^2}{16} + \frac{3x}{32} + \frac{1}{64x} + \frac{1}{4}$

Thus $\lim_{n \to +\infty} U(P, f) = \lim_{n \to +\infty} \frac{n^3}{64} + \frac{n^2}{16} + \frac{3x}{32} + \frac{1}{64x} + \frac{1}{4} = \frac{1}{4} = \int_0^1 (x^3)dx$.

d.)$\int_a^b x^3 dx =$

We know from the previous exercise that $\int_a^b f = \lim_{n \to +\infty} U(P, f)$.

$\Delta x = \frac{a-b}{n}$

Let $P_n = \{\frac{k(a-b)}{n} : k = a, b, \cdots, n\}$.Thus,

$U(P_n, f) = \sum_{k=1}^{n} f(\frac{k(a-b)}{n})(\frac{(a-b)}{n}) = \sum_{k=1}^{n} (\frac{k(a-b)}{n})^4 (\frac{(a-b)}{n}) =$
$(\frac{(a-b)}{n})^5 \sum_{k=1}^{n} k^4 = (\frac{(a-b)}{n})^5 * (\frac{n}{2}(n+1))^4 = (a-b)^5 * (\frac{1}{4n^5} * (\frac{n}{2}(n+1))^4 =$
$\frac{n^3}{64} + \frac{n^2}{16} + \frac{3x}{32} + \frac{1}{64x} + \frac{1}{4})$
Thus $\lim_{n \to +\infty} U(P, f) = \lim_{n \to +\infty} (a-b)^5 (\frac{n^3}{64} + \frac{n^2}{16} + \frac{3x}{32} + \frac{1}{64x} + \frac{1}{4}) = \frac{(a-b)^5}{4} =$
$\int_a^b (x^3)dx$.