DISCRETE OPTIMIZATION

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Executive Summary

Homework 2: p. 708–12-21, 12-26 (b,d,f), 12-28, 12-31, 12-41. Due Friday, February 6.

Problems taken from Optimization in Operations Research, Ronald L. Rardin, Prentice Hall. Some definitions are quoted directly.

12.21

Consider the generalized assignment model

$$\begin{aligned}
\min\ & 15x_{1,1} + 10x_{1,2} + 30x_{2,1} + 20x_{2,2} \\
\text{s.t. } & x_{1,1} + x_{1,2} = 1 \\
& x_{2,1} + x_{2,2} = 1 \\
& 30x_{1,1} + 50x_{2,1} \leq 80 \\
& 30x_{1,2} + 50x_{2,2} \leq 60 \\
& x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2} = 0 \text{ or } 1
\end{aligned}$$

Solve with $\boldsymbol{v} = (0,0), \boldsymbol{v} = (10,12)$, *and* $\boldsymbol{v} = (200,100)$.

a.) *Use total enumeration to compute an optimal solution.*

Since we have four binary variables that there are a total of $2^4 = 16$ possible solutions. However, not all of these solutions are feasible to the ILP. In consideration of the size of the problem, calculations to determine feasibility and objective values were performed in excel. A concise table of the results is available in Appendix A.2 on page 7.

These calculations let us to conclude that there were three feasible solutions. Theses solutions and their corresponding objective values are as follows

Solution $(x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2})$	Objective Value
(1,0,1,0)	45
(0,1,1,0)	40
(1,0,0,1)	35

Therefore we see through total enumeration that the optimal objective value is 35 with a corresponding optimal solution of $(1, 0, 0.1)$.

b.) *Form a LaGrangian relaxation dualizing the first and second constraints with LaGrange multipliers* v_1 *and* v_2.

$$\begin{aligned}
\min\ & 15x_{1,1} + 10x_{1,2} + 30x_{2,1} + 20x_{2,2} \\
& + v_1(1 - x_{1,1} - x_{1,2}) + v_2(1 - x_{2,1} - x_{2,2}) \\
\text{s.t. } & 30x_{1,1} + 50x_{2,1} \leq 80 \\
& 30x_{1,2} + 50x_{2,2} \leq 60 \\
& x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2} = 0 \text{ or } 1
\end{aligned}$$

c.) *Explain how the dualization in part* ***b.)*** *leaves a relaxation that is easier to solve than the full ILP.*

This dualization relaxes the two main linearity constraints of the model while maintaining the integrality constraints. The relaxation is easier to solve than the full ILP because after dualization each $x_{i,j}$ occurs in exactly one remaining constraint. This makes the relaxation tractable as it could be thus solved as a series of two simple Knapsack problems, which are significantly easier to solve than other ILPs.

d.) *Use total enumeration to solve the LaGrangian relaxation of part **b.)** with $v_1 = v_2 = 0$, and verify that the relaxation optimal value provides a lower bound on the true optimal value computed in part **a.)**.*

If $v_1, v_2 = 0$, then we are left with the following dualized LaGrangian relaxation problem.

$$\begin{aligned} \min\ & 15x_{1,1} + 10x_{1,2} + 30x_{2,1} + 20x_{2,2} \\ \text{s.t. } & 30x_{1,1} + 50x_{2,1} \le 80 \\ & 30x_{1,2} + 50x_{2,2} \le 60 \\ & x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2} = 0 \text{ or } 1 \end{aligned}$$

We are able to once again utilize excel to totally enumerate and check the feasibility of solutions. As a result of this relaxation, we have found a solution set that comprises of 12 feasible solutions. The objective function values for these twelve solutions were then calculated.

The optimal objective value and solution were found to be 0 and $(0, 0, 0, 0)$ respectively. The table containing the results of our calculations is available in Appendix A.3 on page 7.

We can verify that this solution is a viable lower bound on the true optimal of 35 by simply comparing the two; since $0 \le 35$, we know that 0 is a lower bound on 35 is thus viable.

Note that this optimal solution isn't particularly useful for a general assignment problem as it represents the action set of no action. It is a good illustration of the necessity of choosing appropriate LaGrangian values. While this might be the cheapest option, this isn't necessarily a good option for the original ILP.

e.) *Repeat part **d.)** with $\boldsymbol{v} = (10, 12)$.* The dualized LaGrangian relaxation is as follows

$$\begin{aligned} \min\ & 15x_{1,1} + 10x_{1,2} + 30x_{2,1} + 20x_{2,2} \\ & + 10(1 - x_{1,1} - x_{1,2}) + 12(1 - x_{2,1} - x_{2,2}) \\ \text{s.t. } & 30x_{1,1} + 50x_{2,1} \le 80 \\ & 30x_{1,2} + 50x_{2,2} \le 60 \\ & x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2} = 0 \text{ or } 1 \end{aligned}$$

Once again, we utilize excel to enumerate the solutions, check constraints, and calculate objective values. The resultant table is available in Appendix A.4 on 8.

In this case there were two optimal solutions: $(0, 0, 0, 0)$ and $(0, 1, 0, 0)$ each with an optimal objective value of 22. As before, we know that this relaxed optimal is a lower bound on the ILP since it is less than the true optimal, i.e. $22 \le 35$. This is a much better lower bound on the true optimum than the previous and thus might be considered useful.

f.) *Repeat part **d.)** with $\boldsymbol{v} = (200, 100)$.* The dualized LaGrangian relaxation is as follows

$$\begin{aligned} \min\ & 15x_{1,1} + 10x_{1,2} + 30x_{2,1} + 20x_{2,2} \\ & + 200(1 - x_{1,1} - x_{1,2}) + 100(1 - x_{2,1} - x_{2,2}) \\ \text{s.t. } & 30x_{1,1} + 50x_{2,1} \le 80 \\ & 30x_{1,2} + 50x_{2,2} \le 60 \\ & x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2} = 0 \text{ or } 1 \end{aligned}$$

As before, excel was utilized to make the calculations ant the resultant table is available in Appendix A.5 on 8.

We have an optimal objective value of -145, which is an extreme, but viable, lower bound on the true optimal of 35 as $-145 \le 35$. The corresponding optimal solution is $(1, 1, 1, 0)$. This is another case which illustrates the necessity of choosing LaGrangian values wisely. Since we chose values that were too large, we received a lower bound that is not particularly useful.

12.26

Suppose that a minimizing ILP is being solved by LP-based branch and bound Algorithm 12A over decision variables $x_1, x_2, x_3 = 0$ or 1, $x_4 \geq 0$. Show how the search should process the node with $x_2 = 1$ and other variables free if the corresponding LP relaxation has each of the following outcomes. Assume that the incumbent solution value is 100.

Note: Algorithm 12A is included in appendix B on page 9 for reference.

Let us restate for clarity:

$$\begin{aligned} &\min v \\ &\text{s.t. constraints} \\ &\quad x_1, x_2, x_3 = 0 \\ &\quad x_4 \geq 0 \end{aligned}$$

Fixed: $x_2 = 1$
Incumbent: $v^* = 100$.

b.) $\widetilde{x} = (0.2, 1, 0.77, 4.5)$, value $\widetilde{v} = 116$.

We know that the objective value of the incumbent is the best feasible value for the original problem and is an upper bound on the optimal solution since this is a minimization problem. Therefore, since the node objective value of 116 is greater than the incumbent value of 100, we can fathom by bounds and terminate the branch.

d.) LP Relaxation infeasible.

If the LP relaxation itself is infeasible, then the current candidate problem must also be infeasible, meaning that we may fathom by bounds and terminate the branch.

f.) $\widetilde{x} = (0.4, 1, 0.1, 5.9)$, value $\widetilde{v} = 83$.

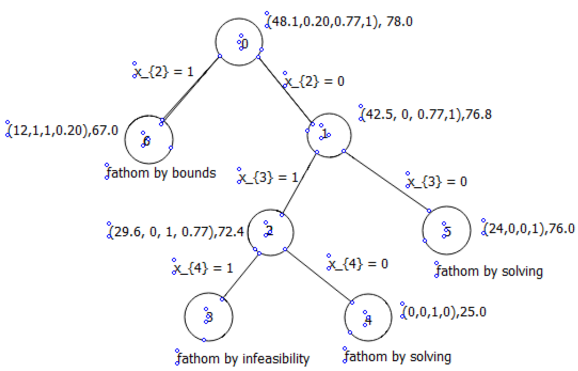
In this case, the optimal objective value of 83 is less than the incumbent value of 100. Therefore, we would continue to branch from this candidate problem. Since both x_1 and x_3 have integrality constraints, we have two options for branching. Either of the selections are valid and the choice may be made heuristically. If we selected x_1, we would reintroduce the corresponding integrality constraint and branch on it. This results in child nodes with x_1 fixed to either 1 or 0. We may similarly proceed with x_3 and create child nodes with x_3 fixed to either 1 or 0.

12.28

The following table shows the LP relaxation outcomes for all possible combinations of fixed and free variables in branch and bound solution of a minimizing integer linear program over decision variables $x_1 \geq 0$, and $x_2, x_3, x_4 = 0$ or 1.

x_2	x_3	x_4	$\widetilde{x}$	$\widetilde{v}$
#	#	#	$(48.1, 0.20, 0.77, 1)$	78.0
#	#	0	$(0, 1, 1, 0)$	59.0
#	#	1	$(48.1, 0.20, 0.77, 1)$	78.0
#	0	#	$(29.6, 0.20, 0, 1)$	77.2
#	0	0	$(0, 1, 0, 0)$	34.0
#	0	1	$(29.6, 0.20, 0, 1)$	77.2
#	1	#	$(41.6, 0.43, 1, 0.77)$	75.0
#	1	0	$(0, 1, 1, 0)$	59.0
#	1	1	*Infeasible*	–
0	#	#	$(42.5, 0, 0.77, 1)$	76.8
0	#	0	$(0, 0, 1, 0)$	25.0
0	#	1	$(42.5, 0, 0.77, 1)$	76.8
0	0	#	$(24, 0, 0, 1)$	76.0
0	0	0	$(0, 0, 0, 0)$	0.0
0	0	1	$(24, 0, 0, 1)$	76.0
0	1	#	$(29.6, 0, 1, 0.77)$	72.4
0	1	0	$(0, 0, 1, 0)$	25.0
0	1	1	*Infeasible*	–
1	#	#	$(12, 1, 1, 0.20)$	67.0
1	#	0	$(0, 1, 1, 0)$	59.0
1	#	1	*Infeasible*	–
1	0	#	$(0, 1, 0, 0.20)$	54.0
1	0	0	$(0, 1, 1, 0)$	34.0
1	0	1	*Infeasible*	–
1	1	#	$(12, 1, 1, 0.20)$	67.0
1	1	0	$(0, 1, 1, 0)$	59.0
1	1	1	*Infeasible*	–

Solve the problem by LP-based Algorithm 12A and record your results in a branch and bound *tree. Apply the depth first rule for selecting among active nodes and pick whichever of = 0 and = 1 is closest to the preceding relaxation value when nodes have equal depth. Branch on the integer-restricted variable with fractional relaxation value nearest to integer.*



We begin the algorithm by assuming that there is no initial incumbent value since it wasn't specified. Next, we will initialize our algorithm with the integrality constraints relaxed on all variables. This is a solution of $(48.1, 0.20, 0.77, 1)$ with an objective value of 78.0.

We see that while all three variables with integrality constraints are currently non-integer, x_2 has a value of 0.20 and is thus closest to an integer value. Therefore, we will branch on x_2 and evaluate $x_2 = 0$ first.

Now we have a candidate problem solution of $(42.5, 0, 0.77, 1)$ with an objective value of 76.8. This value doesn't violate any bounds and isn't an integer solution, therefore we may continue the algorithm. Since x_3 is closest of the currently unconstrained variables to an integer value (and it is closest to the value of 1), we will branch of x_3 and explore the branch $x_3 = 1$ first.

The solution to our new candidate problem is $(29.6, 0, 1, 0.77)$ with an objective value of 72.4. Here the only remaining variable which need to meet integrality conditions is x_4. The solution value for that variable is closest to 0.77, so we will branch first on $x_4 = 1$.

However, when we look to the provided table for the corresponding solution, we see that it is infeasible to have $x_4 = 1$ when $x_2 = 0$ and $x_3 = 1$. Therefore, we will fathom by infeasibility and terminate the branch.

We will now explore the $x_4 = 0$ branch. The corresponding solution to the candidate problem is $(0, 0, 1, 0)$ and has an objective value of 25.0. This is our first feasible integer solution and objective value, therefore we will update our incumbent and fathom by solving.

We must now select another unexplored node The deepest eligible node is the branch corresponding to $x_3 = 0$. Therefore, we will obtain the following integer solution $(24, 0, 0, 1)$ with corresponding objective value 76.0. This is greater than our current incumbent value of 25.0. Thss represents a new incumbent and lower bound. Therefore we may update our incumbent objective value to 76.0.

We have one final unexplored node and it is the candidate problem corresponding to the branch $x_2 = 1$. This gives us a solution of $(12, 1, 1, 0.20)$ and an objective value of 67.0. This solution's objective value is less than our current incumbent value of 76.0. Thus we can fathom by bounds and terminate the branch.

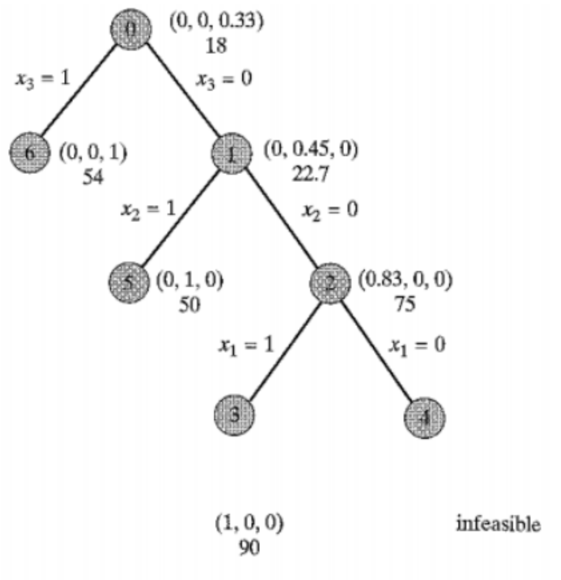
At this point we have either fully explored or terminated every branch in our tree. Therefore, our current incumbent solution of $(24, 0, 0, 1)$ with a corresponding optimal objective value of 76.0 is optimal to the original ILP.

12.31

The branch and bound tree that follows records solution of the knapsack model

$$\begin{aligned} \min\ & 90x_1 + 50x_2 + 54x_3 \\ \text{s.t.}\ & 60x_1 + 110x_2 + 150x_3 \geq 50 \\ & x_1, x_2, x_3 = 0 \text{ or } 1 \end{aligned}$$

by LP-based Algorithm 12A under the rules of Exercise 12-27.



Briefly describe the processing including how and why nodes are branched or terminated, when incumbent solutions were discovered, and what solution proved optimal. Assume that there was no initial incumbent solution.

We begin by relaxing all integrality constraints on the variables x_1 to x_3. This gives us the solution $(0, 0, .33)$ and corresponding objective value of 18. We will branch on x_3 since it has the only non-integer value in the solution. We will first explore the branch corresponding to constraint $x_3 = 0$ as 0.33 is closer to 0 than it is to 1.

The solution to this new candidate problem is $(0, 0.45, 0)$ with a corresponding objective value of 22.7. Since x_2 is non integer in the solution with a value of 0.45, we will branch on the integrality of x_2. Furthermore, since its value is closest to 0, we will explore the branch corresponding to the constraint $x_2 = 0$ first.

We thus arrive at the solution $(0.83, 0, 0)$ and corresponding objective value 75. Since the solution is non integer, we will branch on the only viable candidate that remains, x_1. We will explore the branch corresponding to $x_1 = 1$ first since its solution value, 0.83, is closer to one.

This leads us to the integer solution of $(1, 0, 0)$ with the associated objective value of 90. Since we do not currently have an incumbent, this objective value automatically becomes our incumbent and we have an upper bound of 90. This branch is now terminated as it has been fathomed by solving.

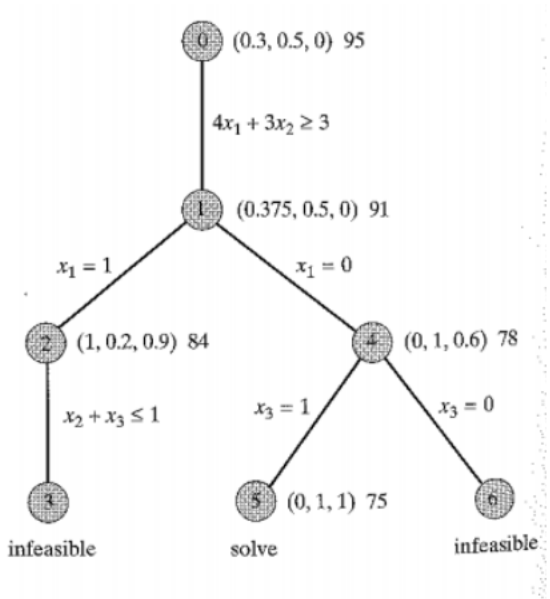
We have fully explored this branch and must now look for the deepest branch that is not yet explored. That branch corresponds with the $x_2 = 1$ constraint that is off of node 1. This gives us the integer solution of $(0, 1, 0)$ and a corresponding objective value 50. This is less than our current incumbent of 90. Thus, we update our incumbent solution and objective value so that 50 becomes the new upper bound on the optimum. This branch is now fathomed by solving and terminated.

There is only one branch that remains to be explored and that is the one corresponding to $x_3 = 1$ that is coming off of node 0. This corresponds to an integer solution of $(0, 0, 1)$ with an objective value of 54. This objective value is larger than our current incumbent, however, and we will therefore fathom by bounds and terminate this branch.

We have fully explored the tree and the algorithm now terminates. Therefore our current incumbent is an optimal integer solution of $(0, 1, 0)$ that is feasible to the original problem and has an optimal objective value of 50.

12.41

The following tree records the solution of a maximizing ILP over $x_1, x_2, x_3 = 0$ or 1 by branch and cut Algorithm 12B. LP relaxations solutions are shown next to each node.



Briefly describe the processing, including how and why nodes were branched tightened or terminated when incumbent solutions were discovered, and what solution proved optimal. Assume that all added inequalities are valid for the original ILP.

Note: Algorithm 12B is included in appendix C on page 10 for reference.

We initialize at node 0 with all integer variables unconstrained and the corresponding non-integer solution $(0.3, 0.5, 0)$ and objective value of 95. We then introduce the valid inequality $4x_1 + 3x_2 \geq 3$ to reduce the search space of the relaxation.

This results in the non-integer solution $(0.375, 0.5, 0)$ with a corresponding objective value of 91. There are no valid inequalities that we can introduce to reduce the search space at this time, therefore we must branch on the integrality constraints of one of the integer variables. There are such variables with non-integer solution values: x_1 and x_2. We will branch on the integrality constraint of $x_1 = 0$ since the solution value is closer to an integer value than x_2 and that value is 0.

We therefore arrive at the fractional solution $(1, 0.2, 0.9)$ with objective value 84. Now we add the valid inequality $x_2 + x_3 \leq 1$ to reduce the search space. This however results in an infeasible candidate problem. Thus we fathom by infeasibility and terminate the branch.

Now we will explore the only open branch, the branch corresponding to the integrality constraint $x_1 = 0$. This results in the non-integer solution $(0, 1, 0.6)$ and corresponding objective value 78. We have no further valid inequalities to introduce at this time and will instead branch on the integrality constraint of the only viable candidate, x_3. We will select the branch corresponding to $x_3 = 1$ as the solution value is closest to 1.

This leads us to our first integer solution of $(0, 1, 1)$ with a corresponding objective value of 75. It becomes our incumbent value and is a minimum bound on our optimum. We now fathom by solving and terminate this branch.

There is one branch that we have yet to explore and that is the branch associated with $x_3 = 0$. This integrality constraint results in an infeasible solution and thus we may fathom by infeasibility and terminate the branch.

We have completed the algorithm and have no more branches to explore. Therefore, we conclude that the incumbent solution $(0, 1, 1)$ with corresponding objective value 75 is the optimum.

A Problem 12-21

A.1

General Assignment Problem

$$\begin{aligned} \min\ & 15x_{1,1} + 10x_{1,2} + 30x_{2,1} + 20x_{2,2} \\ \text{s.t.}\ & x_{1,1} + x_{1,2} = 1 \\ & x_{2,1} + x_{2,2} = 1 \\ & 30x_{1,1} + 50x_{2,1} \leq 80 \\ & 30x_{1,2} + 50x_{2,2} \leq 60 \\ & x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2} = 0 \text{ or } 1 \end{aligned}$$

A.2 a.)

$x_{1,1}$	$x_{1,2}$	$x_{2,1}$	$x_{2,2}$	**Constraint 1**	**Constraint 2**	**Constraint 3**	**Constraint 4**	**FEASIBLE**
0	0	0	0	FALSE	FALSE	TRUE	TRUE	FALSE
1	0	0	0	TRUE	FALSE	TRUE	TRUE	FALSE
0	1	0	0	TRUE	FALSE	TRUE	TRUE	FALSE
1	1	0	0	FALSE	FALSE	TRUE	TRUE	FALSE
0	0	1	0	FALSE	TRUE	TRUE	TRUE	FALSE
1	0	1	0	TRUE	TRUE	TRUE	TRUE	TRUE
0	1	1	0	TRUE	TRUE	TRUE	TRUE	TRUE
1	1	1	0	FALSE	TRUE	TRUE	TRUE	FALSE
0	0	0	1	FALSE	TRUE	TRUE	TRUE	FALSE
1	**0**	**0**	**1**	**TRUE**	**TRUE**	**TRUE**	**TRUE**	**TRUE**
0	1	0	1	TRUE	TRUE	TRUE	FALSE	FALSE
1	1	0	1	FALSE	TRUE	TRUE	FALSE	FALSE
0	0	1	1	FALSE	FALSE	TRUE	TRUE	FALSE
1	0	1	1	TRUE	FALSE	TRUE	TRUE	FALSE
0	1	1	1	TRUE	FALSE	TRUE	FALSE	FALSE
1	1	1	1	FALSE	FALSE	TRUE	FALSE	FALSE

A.3 d.)

$x_{1,1}$	$x_{1,2}$	$x_{2,1}$	$x_{2,2}$	$30x_{1,1} + 50x_{2,1} \leq 80$	$30x_{1,2} + 50x_{2,2} \leq 60$	**FEASIBLE**	**Objective Value**
0	**0**	**0**	**0**	**TRUE**	**TRUE**	**TRUE**	**0**
1	0	0	0	TRUE	TRUE	TRUE	15
0	1	0	0	TRUE	TRUE	TRUE	10
1	1	0	0	TRUE	TRUE	TRUE	25
0	0	1	0	TRUE	TRUE	TRUE	30
1	0	1	0	TRUE	TRUE	TRUE	45
0	1	1	0	TRUE	TRUE	TRUE	40
1	1	1	0	TRUE	TRUE	TRUE	55

0	0	0	1	TRUE	TRUE	TRUE	20
1	0	0	1	TRUE	TRUE	TRUE	35
0	1	0	1	TRUE	FALSE	FALSE	FALSE
1	1	0	1	TRUE	FALSE	FALSE	FALSE
0	0	1	1	TRUE	TRUE	TRUE	50
1	0	1	1	TRUE	TRUE	TRUE	65
0	1	1	1	TRUE	FALSE	FALSE	FALSE
1	1	1	1	TRUE	FALSE	FALSE	FALSE

A.4 e.)

$x_{1,1}$	$x_{1,2}$	$x_{2,1}$	$x_{2,2}$	$30x_{1,1} + 50x_{2,1} \leq 80$	$30x_{1,2} + 50x_{2,2} \leq 60$	**FEASIBLE**	**Objective Value**
0	**0**	**0**	**0**	**TRUE**	**TRUE**	**TRUE**	**22**
1	0	0	0	TRUE	TRUE	TRUE	27
0	**1**	**0**	**0**	**TRUE**	**TRUE**	**TRUE**	**22**
1	1	0	0	TRUE	TRUE	TRUE	27
0	0	1	0	TRUE	TRUE	TRUE	40
1	0	1	0	TRUE	TRUE	TRUE	45
0	1	1	0	TRUE	TRUE	TRUE	40
1	1	1	0	TRUE	TRUE	TRUE	45
0	0	0	1	TRUE	TRUE	TRUE	30
1	0	0	1	TRUE	TRUE	TRUE	35
0	1	0	1	TRUE	FALSE	FALSE	FALSE
1	1	0	1	TRUE	FALSE	FALSE	FALSE
0	0	1	1	TRUE	TRUE	TRUE	48
1	0	1	1	TRUE	TRUE	TRUE	53
0	1	1	1	TRUE	FALSE	FALSE	FALSE
1	1	1	1	TRUE	FALSE	FALSE	FALSE

A.5 f.)

$x_{1,1}$	$x_{1,2}$	$x_{2,1}$	$x_{2,2}$	$30x_{1,1} + 50x_{2,1} \leq 80$	$30x_{1,2} + 50x_{2,2} \leq 60$	**FEASIBLE**	**Objective Value**
0	0	0	0	TRUE	TRUE	TRUE	300
1	0	0	0	TRUE	TRUE	TRUE	115
0	1	0	0	TRUE	TRUE	TRUE	110
1	1	0	0	TRUE	TRUE	TRUE	-75
0	0	1	0	TRUE	TRUE	TRUE	230
1	0	1	0	TRUE	TRUE	TRUE	45
0	1	1	0	TRUE	TRUE	TRUE	40
1	**1**	**1**	**0**	**TRUE**	**TRUE**	**TRUE**	**-145**
0	0	0	1	TRUE	TRUE	TRUE	220
1	0	0	1	TRUE	TRUE	TRUE	35
0	1	0	1	TRUE	FALSE	FALSE	FALSE

1	1	0	1	TRUE	FALSE	FALSE	FALSE
0	0	1	1	TRUE	TRUE	TRUE	150
1	0	1	1	TRUE	TRUE	TRUE	-35
0	1	1	1	TRUE	FALSE	FALSE	FALSE
1	1	1	1	TRUE	FALSE	FALSE	FALSE

B Algorithm 12A

ALGORITHM 12A: LP-BASED BRANCH AND BOUND (0–1 ILPS)

Step 0: Initialization. Make the only active partial solution the one with all discrete variables free, and initialize solution index $t \leftarrow 0$. If any feasible solutions are known for the model, also choose the best as incumbent solution $\hat{\mathbf{x}}$ with objective value $\hat{v}$. Otherwise, set $\hat{v} \leftarrow -\infty$ if the model maximizes and $\hat{v} \leftarrow +\infty$ if it minimizes.

Step 1: Stopping. If active partial solutions remain, select one as $\mathbf{x}^{(t)}$, and proceed to Step 2. Otherwise, stop. If there is an incumbent solution $\hat{\mathbf{x}}$, it is optimal, and if not, the model is infeasible.

Step 2: Relaxation. Attempt to solve the linear programming relaxation of the candidate problem corresponding to $\mathbf{x}^{(t)}$.

Step 3: Termination by Infeasibility. If the LP relaxation proved infeasible, there are no feasible completions of partial solution $\mathbf{x}^{(t)}$. Terminate $\mathbf{x}^{(t)}$, increment $t \leftarrow t+1$, and return to Step 1.

Step 4: Termination by Bound. If the model maximizes and LP relaxation optimal value $\tilde{v}$ satisfies $\tilde{v} \leq \hat{v}$, or it minimizes and $\tilde{v} \geq \hat{v}$, the best feasible completion of partial solution $\mathbf{x}^{(t)}$ cannot improve on the incumbent. Terminate $\mathbf{x}^{(t)}$, increment $t \leftarrow t+1$, and return to Step 1.

Step 5: Termination by Solving. If the LP relaxation optimum $\tilde{\mathbf{x}}^{(t)}$ satisfies all binary constraints of the model, it provides the best feasible completion of partial solution $\mathbf{x}^{(t)}$. After saving it as new incumbent solution

$$\hat{\mathbf{x}} \leftarrow \tilde{\mathbf{x}}^{(t)}$$
$$\hat{v} \leftarrow \tilde{v}$$

terminate $\mathbf{x}^{(t)}$, increment $t \leftarrow t+1$, and return to Step 1.

Step 6: Branching. Choose some free binary-restricted component x_p that was fractional in the LP relaxation optimum, and branch $\mathbf{x}^{(t)}$ by creating two new actives. One is identical to $\mathbf{x}^{(t)}$ except that x_p is fixed $= 0$, and the other the same except that x_p is fixed $= 1$. Then increment $t \leftarrow t+1$ and return to Step 1.

C Algorithm 12B

ALGORITHM 12B: BRANCH AND CUT (0-1 ILP'S)

Step 0: Initialization. Make the only active partial solution the one with all discrete variables free, and initialize solution index $t \leftarrow 0$. If any feasible solutions are known for the model, also choose the best as incumbent solution $\hat{\mathbf{x}}$ with objective value $\hat{\nu}$. Otherwise, set $\hat{\nu} \leftarrow -\infty$ if the model maximizes and $\hat{\nu} \leftarrow +\infty$ if it minimizes.

Step 1: Stopping. If active partial solutions remain, select one as $\mathbf{x}^{(t)}$, and proceed to Step 2. Otherwise, stop. If there is an incumbent solution $\hat{\mathbf{x}}$, it is optimal, and if not, the model is infeasible.

Step 2: Relaxation. Attempt to solve the linear programming relaxation of the candidate problem corresponding to $\mathbf{x}^{(t)}$.

Step 3: Termination by Infeasibility. If the LP relaxation proved infeasible, there are no feasible completions of partial solution $\mathbf{x}^{(t)}$. Terminate $\mathbf{x}^{(t)}$, increment $t \leftarrow t+1$, and return to Step 1.

Step 4: Termination by Bound. If the model maximizes and LP relaxation optimal value $\tilde{\nu}$ satisfies $\tilde{\nu} \leq \hat{\nu}$, or it minimizes and $\tilde{\nu} \geq \hat{\nu}$, the best feasible completion of partial solution $\mathbf{x}^{(t)}$ cannot improve on the incumbent. Terminate $\mathbf{x}^{(t)}$, increment $t \leftarrow t+1$, and return to Step 1.

Step 5: Termination by Solving. If the LP relaxation optimum $\tilde{\mathbf{x}}^{(t)}$ satisfies all binary constraints of the model, it provides the best feasible completion of partial solution $\mathbf{x}^{(t)}$. After saving it as new incumbent solution by $\hat{\mathbf{x}} \leftarrow \tilde{\mathbf{x}}^{(t)}$ and $\hat{\nu} \leftarrow \tilde{\nu}$, terminate $\mathbf{x}^{(t)}$, increment $t \leftarrow t+1$, and return to Step 1.

Step 6: Valid Inequality. Attempt to identify a valid inequality for the full ILP model that is violated by the current relaxation optimum $\tilde{\mathbf{x}}^{(t)}$. If successful, make the constraint a part of the full model increment $t \leftarrow t+1$, and return to Step 2.

Step 7: Branching. Choose some free binary-restricted component x_p that was fractional in the last LP relaxation optimum, and branch $\mathbf{x}^{(t)}$ by creating two new actives. One is identical to $\mathbf{x}^{(t)}$ except that x_p is fixed $= 0$, and the other the same except that x_p is fixed $= 1$. Then increment $t \leftarrow t+1$ and return to Step 1.