Advanced Calculus II
Unit 6.1: Problems 14, 15, 17, 19
Unit 6.2: 2a, 4, 6, 7b

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6.1.14a

If $f \in R[a,b]$, prove directly (without using Theorems 6.1.9 or 6.1.13 that $|f| \in R[a,b]$.

We know that $f \in R[a,b]$ implies that for all $\epsilon \geq 0$, there exists a partition P of $[a,b]$ such that
$U(P,f) - L(P,f) < \epsilon$ where $P = \{x_0, x_1, \cdots, x_n\}$.

$M_i = \sup\{f(t) | x_{i-1} \leq t \leq x_i\}$

$m_i = \inf\{f(t) | x_{i-1} \leq t \leq x_i\}$

$M_i^* = \sup\{|f(t)| | x_{i-1} \leq t \leq x_i\}$

$m_i^* = \inf\{|f(t) | x_{i-1} \leq t \leq x_i\}$

If $M_i, m_i < 0$ or $M_i, m_i > 0$, then $M_i - m_i = M_i^* - m_i^*$.

If $M_i \geq 0$ and $m_i \leq 0$, then $M_i^* - m_i^* \leq M_i - m_i$.

Therefore, $U(P,|f|) - L(P,|f|) \leq U(P,f) - L(P,f) < \epsilon$.

Therefore, because we know that a bounded, real-valued function f is Reiemann integrable on $[a,b]$ if an only if for every $\epsilon > 0$, there exists a partition P of $[a,b]$ such that $U(P,f) - L(P,f) < epsilon$, we can say that $|F| \in R[a,b]$.

6.1.14b

If $|f|\ inR[a,b]$, is $f \in R[a,b]$?

$|f| \in R[a,b]$ does not necessarily imply $f \in R[a,b]$.
A good example would be

$$f(x) = \begin{cases} 1 & x = \frac{1}{n}, \forall n \in N \\ -1 & otherwise \end{cases}$$

Thus, $|f| \in R[a,b]$ but f is not (too much discontinuity).

6.1.15a

If $f \in R[a,b]$, prove directly that $f^2 \in [a,b]$.

Let ϵ>0. Since $f \in B[a,b]$, there exists $M \in R$ such that $|f(x)| \leq M$ for all $x \in [a,b]$.

Let P be a partition of $[a,b]$ with $P = x_0, x_1, ..., x_n$, such that $U(P,f) - L(P,f) < \frac{\epsilon}{2M}$.
$|f^2(x_i) - f^2(x_{i-1})| = |f(x_i) + f(x_{i-1})||f(x_i) - f(x_{i-1})|$
$\leq (|f(x_i)| + |f(x_{i-1})|)|f(x_i) - f(x_{i-1})| < 2M|f(x_i) - f(x_{i-1})|$, for all $x_i, x_{i-1} \in [a,b]$.

$U(P,f^2) - L(P,f^2) = \sum_{i=1}^{n} |f^2(x_i) - f^2(x_{i-1}|\Delta x_i$
$\leq \sum_{i=1}^{n} 2M|f(x_i) - f(x_{i-1})|\Delta x_i = 2M(U(P,f) - L(P,f)) < 2M\frac{\epsilon}{2M} = \epsilon$

Therefore, if $f \in R[a,b]$, $f^2 \in [a,b]$.

6.1.15b

Give an example of a bounded function f on [a,b] for which $f^2 \in R[a,b]$, but $f \notin R[a,b]$

A good example would be $f : [a,b] \to \mathbb{R}$,

$$f(x) = \begin{cases} 1 & x \in Q \\ -1 & otherwise \end{cases}$$

$f^2(x) = 1$ for all $x \in [a,b]$ and is thus Riemann integrable, but $f(x)$ is not Riemann integrable because it contains too many points of discontinuity.

6.1.17a

If E has measure zero, prove that every subset of E has measure zero.

$E \subset \mathbb{R}$ has measure zero implies that for all $\epsilon > 0$, there exists a finite or countable collection$\{I_n\}_n$ of open intervals such that $E \subset \cup_n I_n$ and $\sum_n l(I_n) < \epsilon$, where $l(I_n)$ represents the length of interval I_n.

For any $F \subset E$, $F \subset E \subset \cup_n I_n$ and $\sum l(I_n) < \epsilon$.

This implies that $F \subset \cup_n I_n$ and $\sum l(I_n) < \epsilon$, which is the definition of a set with measure zero.

Therefore, for any set $E \subset \mathbb{R}$ with measure zero, any subset $F \subset E$ will also have measure zero.

6.17.b

If E_1, E_2 have measure zero, prove that $E_1 \cup E_2 = E^*$ has measure zero.

E_1 has measure zero implies that if given any $\epsilon \geq 0$, there exists a finite or countable collection $\{J_m\}_m$ of open intervals such that $E \subset \cup_m J_m$ and $\sum l(J_m) < \epsilon_2$, where $l(J_m)$ denotes the length of interval J_m.

E_2 has measure zero implies that if given any $\epsilon \geq 0$, there exists a finite or countable collection $\{I_n\}_n$ of open intervals such that $E \subset \cup_n I_n$ and $\sum l(I_n) < \epsilon_2$, where $l(I_n)$ denotes the length of interval I_n.

$E_1 \cup E_2 = E^* \subset (\cup_m J_m) \cup (\cup_n I_n) = \cup_k L_k$

$\sum l(L_k) \leq \sum l(J_m) + \sum (I_n) \leq \epsilon_1 + \epsilon_2 = \epsilon.$

Therefore, for any $\epsilon = \epsilon_1 + \epsilon_2 \geq 0$, there exists a finite or countable collection $\{L_k\}_k$ of open intervals such that
$E_1 \cup E_2 = E^* \subset \cup_k L_k = \cup_m J_m) \cup (\cup_n I_n$ and $\sum l(L_k)\epsilon$ where $l(L_k)$ denotes the length of interval L_k.
Thus, by definition, E^* has measure zero.

6.17.C

If each $E_n, n = 1, 2, \cdots$ *have measure zero, prove that* $\cup_{n=1}^{\infty} E_n$ *has measure zero.*

$E_n, n = 1, 2, \cdots$ has measure zero implies that if given any $\epsilon \geq 0$, there exists a finite or countable collection $\{I_{n,m}\}_m$ of open intervals such that $E \subset \cup_m I_{n,m}$and $\sum_{n=1}^{\infty} l(I_{n,m})<\epsilon_n$, where $l(I_{n,m})$ denotes the length of interval $I_{n,m}$.

$\cup_{n=1}^{\infty} E_n \subset \cup_{n=1}^{\infty} \cup_m I_{n,m} = \cup_k L_k$

$\sum l(L_k) \leq \sum_{n=1}^{\infty} l(I_{n,m})<\epsilon_n$

Therefore, for any $\epsilon = \sum_{n=1}^{\infty} \epsilon_n \geq 0$, there exists a finite or countable collection $\{L_k\}_k = \cup_{n=1}^{\infty} \{I_{n,m}\}_m$ of open intervals such that $E^* \subset \cup_k L_k$ and $\sum_{k=1}^{\infty} l(L_k)<\epsilon$, where $l(L_k)$ denotes the length of interval L_k. Thus, $E^* = \cup_{n=1}^{\infty} E_n$ has measure zer0 by definition.

6.1.19

Prove directly that the function

$$f(x) = \begin{cases} 1 & x = 0 \\ 0, & x \in Q^c \\ \frac{1}{n} & if x = \frac{m}{n} in lowest terms, x \neq 0 \end{cases}$$

is Reimann integrable on $[a, b]$.

By definition, a function is Riemann integrable if for all $\epsilon>0$, there exists a partition P of $[a, b]$ such that $U(P, f) - L(P, f)<\epsilon$.

For any P of $[a, b]$, $L(P, f) = 0$ because $L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i$ where $m_i = \inf\{f(t) : x_{i-1} \leq t \leq x_i\} = 0$.

Therefore, it is sufficient to show thta $U(P, f)<\epsilon$.

Pick and $N \in \mathbb{N}$ such that $\frac{1}{N}<\frac{\epsilon}{2(b-a)}$.

There are only a finite number of $x \in [a, b]$ that are rational with $x = \frac{m}{N}$ for $m \in \mathbb{N}$. We will call this number K_ϵ and define the corresponding set of $\frac{m}{n}$ values as $\{x_i\}_{i=1}^{K_\epsilon} = \{y | y = \frac{m}{N}, m \in \mathbb{N}, y \in [a, b]$.

Pick the partition
$P = \{max[a, x_1 - \frac{\epsilon}{4K_\epsilon}], x_1 + \frac{\epsilon}{4K_\epsilon}, x_2 - \frac{\epsilon}{4K_\epsilon}, x_2 + \frac{\epsilon}{4K_\epsilon}, \cdots, x_2 K_\epsilon - \frac{\epsilon}{4K_\epsilon}, min[x_{K_\epsilon} + \frac{\epsilon}{4K_\epsilon}\}$.

Therefore, $U(P, f) < (2k_\epsilon)\frac{\epsilon}{4K_\epsilon} + \frac{\epsilon(b-a)}{2(b-a)} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Therefore, by Riemann's Criteria for Integrability, $f(x)$ is Riemann integrable on $[a, b]$.

Thus, $f(x)$ is Riemann integrable on $[a, b]$.

6.2.2a

Use the method of Riemann sums to evaluate $\int_a^b x^2 dx$.

$f(x) = x^2$ with $f(x) \in R[a, b]$ implies that $\lim_{||P|| \to a} L(P, f) = \int_a^b x^2 dx$.

Pick $t_i = \frac{1}{3}(x_i^2 + x_i x_{i-1} + x_{i-1}^2)$.

This implies that $L(P, f) = \frac{1}{3}\sum_{i=1}^n (x_i^2 + x_i x_{i-1} + x_{i-1}^2)(x_i - x_{i-1})$
$= \frac{1}{3}\sum_{i=1}^n (x_i^3 - x_{i-1}^3) = \frac{1}{3}(b^3 - a^3)$.

Therefore, $\lim_{||P|| \to \infty} L(P, f) = \frac{1}{3}(b^3 - a^3) = \int_a^b x^2 dx$.

6.2.4a

Let $f \in R[-a, a], a > 0$. *Prove that if* f *is even, (i.e.* $f(-x) = f(x)$ *for all* $x \in [-a, a]$*), then* $\int_{-a}^a f = 2\int_0^a f$.

We know that$f \in R[-a.a]$ implies $\int_{-a}^a f(x)dx = \int_{-a}^0 f(x)dx + \int_0^a f(x)dx =$
$\int_{-a}^0 f(-x)(-1)dx + \int_0^a f(x)dx = -\int_{-a}^0 f(-x)dx + \int_0^a f(x)dx =$
$-\int_{-a}^0 f(x)dx + \int_0^a f(x)dx$

Since $\int_0^a f(x)dx = -\int_{-a}^0 dx$, we make a substitution. Therefore,
$\int_{-a}^a f(x)dx = -\int_{-a}^0 f(x)dx + \int_0^a f(x)dx = \int_0^a f(x)dx + \int_0^a f(x)dx$
$= 2\int_0^a f(x)dx$

Therefore, $\int_- a^a f(x)dx = 2\int_0^a f(x)dx$ if $f(x)$ is even.

6.2.4b

Let $f \in R[-a, a], a>0$*. Prove that if* f *is odd, (i.e.* $f(-x) = -f(x)$ *for all* $x \in [-a, a]$*), then* $\int_{-a}^{a} f = 0$*.*

We know that$f \in R[-a.a]$ implies $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx.$

$\int_{-a}^{0} f(x)dx = \int_{a}^{0} f(-(-x))dx = -\int_{a}^{0}(-f(-x))dx = \int_{a}^{0} f(-x)dx =$
$-\int_{0}^{a} f(-x)dx = -\int_{0}^{a} f(x)dx.$

Which implies $\int_{-a}^{a} f(x)\, dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx = -\int_{0}^{a} f(x)dx + \int_{0}^{a} f(x)dx = 0.$

Therefore, $\int_{-a}^{a} f = 0$, if f is odd.

6.2.6

Let f *be continuous on* $[0.1]$*. Prove that* $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(\frac{k}{n}) = \int_{0}^{1} f(x)dx.$

Since $f \in C[0, 1]$, we know that $f \in R[0, 1]$.

Since $f \in R[0, 1]$, then $\lim_{||P||\to 0} L(P, f)$ exists and
$\lim_{||P||\to 0} L(P, f) = \int_{0}^{1} f(x)dx.$

Since the limit exists for any $t_i \in [x_{i-1}, x_i]$, we can take $t_i = \frac{k}{n}$.

$\lim_{||P||\to 0} L(P, f) = \lim_{||P||\to 0} \sum_{i=1}^{n} f(t_i)\Delta x_i =$
$\lim_{||P||\to 0} \sum_{i=1}^{n} f(t_i)(\frac{1-0}{n}) \lim_{||P||\to 0} \frac{1}{n} \sum_{i=1}^{n} f(t_i) =$
$\lim_{||P||\to 0} \frac{1}{n} \sum_{i=1}^{n} f(\frac{k}{n})$

Therefore, $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(\frac{k}{n}) = \int_{0}^{1} f(x)dx.$

6.2.7b

Use the previous exercise to evaluate the following limit:
$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n^2+k^2}$

We know that $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(\frac{k}{n}) = \int_0^1 f(x)dx$.

From problem 6.2.6, we know that $f(\frac{k}{n}) = \frac{\frac{k}{n}}{1+(\frac{k}{n})^2}$.

Therefore, $f(x) = \frac{x}{1+x^2}$.

$\int_0^1 \frac{x}{1+x^2} = \frac{1}{2}\ln(x^2+1)|_0^1 = \frac{\ln(2)}{2}$
$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n^2+k^2}$.