DISCRETE OPTIMIZATION

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Executive Summary

Homework 3: p. 713–12-45, 12-47, 12-49, 12-51, 12-53, 12-55, 12-57 due Monday, February 16.
Problems taken from Optimization in Operations Research, Ronald L. Rardin, Prentice Hall. Some definitions are quoted directly.

Note: Calculations for the improving search algorithms below are conducted in Excel unless otherwise noted.

12.45

Consider solving (approximately) the ILP.

$$\begin{aligned} \max\ & 12x_1 + 7x_2 + 9x_3 + 8x_4 \\ \text{s.t. } & 3x_1 + x_2 + x_3 + x_4 \leq 3 \\ & x_3 + x_4 \leq 1 \\ & x_1, \ldots, x_4 = 0 \text{ or } 1 \end{aligned}$$

by a version of discrete improving search Algorithm 12C that always advances to the feasible neighbor with the best objective value and uses the single complement neighborhood permitting anyone $x_j = 1$ to be switched to $= 0$ or vice versa.

a.) *Identify a global optimal solution by inspection.* We know from the second constraint that either x_3 or x_4 can equal 1, but not both, as this would violate the constraint. Since they both have the same coefficient in the first constraint, we will thus select the variable with the greatest effect on the objective function, which is x_3. Now, in order to ensure that the second constraint isn't violated but also find optimality, we need to consider the remaining two variables, x_1 and x_2. x_1 has a greater positive impact on the objective function, but assigning x_1 a value of 1 would violate the first constraint. Therefore, we will choose to let $x_2 = 1$ for an optimal solution of $(0, 1, 1, 0)$ and optimal objective value 16.

b.) *Use Algorithm 12C to compute a local optimum starting from $\boldsymbol{x}^{(0)} = (0, 0, 0, 0)$.*

T	$x^{(t)}$	Neighborhood	Value
0	(0,0,0,0)	$(1, 0, 0, 0)$	12
		$(0, 1, 0, 0)$	7
		$(0, 0, 0, 1)$	9
		$(0, 0, 0, 1)$	8
1	$x^{(t)}$	Neighborhood	Value
	(1,0,0,0)	$(0, 0, 0, 0)$	0
		$(1, 1, 0, 0)$	INF
		$(1, 0, 1, 0)$	INF
		$(1, 0, 0, 1)$	INF

Therefore, we see that a local optimum starting from $\mathbf{x}^{(0)} = (0, 0, 0, 0)$ is $\mathbf{x}^{(1)} = (1, 0, 0, 0)$. This is a local optimum because all of the values generated by the move neighborhood from this point are either infeasible or non-improving.

c.) *Apply the multistart extension of the improving search to compute a local optimum by trying starts at $\boldsymbol{x} = (0, 0, 0, 0), (0, 1, 0, 0)$, and $(0, 0, 0, 1)$.* We see that we solved the first search during part (b). We will therefore begin with $\mathbf{x}_2^{(0)} = (0, 1, 0, 0)$.

t	$x^{(t)}$	Neighborhood	Value
0	0 1 0 0	1 1 0 0	INF
		0 0 0 0	0
		0 1 1 0	16
		0 1 0 1	15
t	$x^{(t)}$	Neighborhood	Value
0	0 1 1 0	1 1 1 0	INF
		0 0 1 0	9
		0 1 0 0	7
		0 1 1 1	INF

The improving search algorithm therefore terminates as there is no move in the neighborhood that results in an improving, feasible solution. The local optimum is $\mathbf{x}_2^{(1)} = (0, 1, 1, 0)$ with a local optimal objective value of 16. We will now restart the algorithm at the next starting point, $\mathbf{x}_3^{(0)} = (0, 0, 0, 1)$.

t	$x^{(t)}$	Neighborhood	Value
0	0 0 0 1	1 0 0 1	INF
		0 1 0 1	15
		0 0 1 1	INF
		0 0 0 0	0
1	$x^{(t)}$	Neighborhood	Value
	0 1 0 1	1 1 0 1	INF
		0 0 0 1	8
		0 1 1 1	INF
		0 1 0 0	7

The improving search algorithm therefore terminates as there is no move in the neighborhood that results in an improving, feasible solution. The local optimum is $\mathbf{x}_3^{(1)} = (0, 1, 0, 1)$ with a local optimal objective value of 15.

12.47

Return to the improving search problem of Exercise 12-45.

a.) *Show that $\boldsymbol{x} = (1, 0, 0, 0)$ is a local optimum.* We see from the previous problem that every point in the neighborhood of $(1, 0, 0, 0)$ is either non-improving or infeasible. Therefore, there is no viable move in the move-set and the point is a local optimum.

t	$x^{(t)}$	Neighborhood	Value
0	1 0 0 0	0 0 0 0	0
		1 1 0 0	INF
		1 0 1 0	INF
		1 0 0 1	INF

b.) *Show that if a non-improving move is allowed at $\boldsymbol{x} = (1, 0, 0, 0)$, the next iteration will return this same point.*

If we were to choose the only feasible move in part (a)'s neighborhood, we would select $\mathbf{x}^{(1)} = (0, 0, 0, 0)$, which results in the following move set:

t	$x^{(t)}$	Neighborhood	Value
1	0 0 0 0	1 0 0 0	12
		0 1 0 0	7
		0 0 1 0	9
		0 0 0 1	8
t	$x^{(t)}$	Neighborhood	Value
2	1 0 0 0	0 0 0 0	0
		1 1 0 0	INF
		1 0 1 0	INF
		1 0 0 1	INF

We see that this results in a return to the original point $(1, 0, 0, 0)$ at $t = 2$. Again, there are no improving, feasible solutions and the only non-improving feasible solution is the solution from $t = 1$, which means that we would return to the point $(1, 0, 0, 0)$ if we were again to allow non-improving answers.

12.49

Return to the improving search problem of Exercise 12-45, starting from $\boldsymbol{x}^{(0)} = (1, 0, 0, 0)$. Compute an approximate optimum by tabu search Algorithm 12D, forbidding complementation of a variable for one step after its value changes and limiting the search to $t_{\max} = 5$ moves.

t	$x^{(t)}$	N-hood	Value	Inc.	Tabu
0	1 0 0 0	0 0 0 0	0	12	
		1 1 0 0	INF		
		1 0 1 0	INF		
		1 0 0 1	INF		
t	$x^{(t)}$	N-hood	Value	Inc.	Tabu
1	0 0 0 0			12	(1,0,0,0)
		0 1 0 0	7		
		0 0 1 0	9		
		0 0 0 1	8		
t	$x^{(t)}$	N-hood	Value	Inc.	Tabu
2	0 0 1 0			12	(0,0,1,0)
		1 0 1 0	INF		
		0 1 1 0	16		
		0 0 0 0	0		
t	$x^{(t)}$	N-hood	Value	Inc.	Tabu
3	0 1 1 0			16	(0,0,1,0)
		1 1 1 0	INF		
		0 1 0 0	7		
		0 1 1 1	INF		
t	$x^{(t)}$	N-hood	Value	Inc.	Tabu
4	0 1 0 0			16	(0,0,0,1)
		1 1 0 0	INF		
		0 0 0 0	0		
		0 1 0 1	15		
t	$x^{(t)}$	N-hood	Value	Inc.	Tabu
5	0 1 0 1			16	(0,0,1,0)
		1 1 0 1	INF		
		0 0 0 1	8		
		0 1 0 0	7		

We see that after $t = 5$, we arrive at an incumbent value of 16 corresponding with solution $\hat{\mathbf{x}} = (0, 1, 1, 0)$. This corresponds with one of the local optimums that we found earlier at in Exercise 12-45.

12.51

*Return to the improving search problem of Exercise 12-45 starting from **(0,0,0,1)**. Compute an approximate optimum by simulated annealing Algorithm 12E, using a temperature of $q = 20$, limiting the search to $t_{\max} = 4$ moves and resolving probabilistic decisions with (uniform[0,1]) random numbers* $0.65, 0.10, 0.40, 0.53, 0.33, 0.98, 0.88, 0.37$).

We know from the problem statement given in 12-45 that this problem uses the single complement neighborhood permitting any one $x_j = 1$ to be switched to $= 0$ or vice versa. Therefore, we will divide up the probabilistic space as follows

Probability	Variable Switched
$[0.000, 0.249]$	x_1
$[0.250, 0.499]$	x_2
$[0.500, 0.749]$	x_3
$[0.750, 1.000]$	x_4

We use these probabilistic search values to divide the search space and start the Algorithm.

t	Sol.	Obj. Value	Inc.	q	r	M	δ Obj.	P	O.
0	0001	8	8	20					
1	0011	INF			.65	x_3			
	1001	INF			.10	x_1			
	0101	15		20	.40	x_2	7		A
2	0111	INF	15	20	.53	x_3			
	0001	8			.33	x_2	-7	.70	A
3	0000	0	15	18	.98	x_4	-15	.43	R
	0000	0			.88	x_4	-15	.43	R
	0101	15			.37	x_2	0	1.0	A
4	0101	15	15	16					

The Algorithm terminates at $t = 4$ with an approximate optimal objective solution of $\hat{\mathbf{x}} = 0101$ and a corresponding optimal objective value of 15.

12.53

Return to the improving search problem of Exercise 12-45.

a.) *Show that the solutions* $\boldsymbol{x}^{(1)} = (0, 0, 1, 0)$ *and* $\boldsymbol{x}^{(2)} = (0, 0, 0, 1)$ *are eligible to belong to a genetic algorithm population for the problem.*

We know that in order for the solutions to be eligible to belong to the genetic algorithm population for the problem, they must be feasible for the problem. We know that they are feasible if the satisfy the following constraints:

$$3x_1 + x_2 + x_3 + x_4 \leq 3$$
$$x_3 + x_4 \leq 1$$
$$x_1, \ldots, x_4 = 0 \text{ or } 1$$

We can see that both $\mathbf{x}^{(1)} = (0,0,1,0)$ and $\mathbf{x}^{(2)} = (0,0,0,1)$ are feasible as they don't violate any of the constraints. Therefore, they are eligible.

b.) *Construct all possible crossover results (all cut points for the $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ of part (a).*

Table 1: Genetic Algorithm Crossovers

Crossover #	Parents	Children
1	$\mathbf{x}^{(1)} = 0\vert 010$	$\mathbf{x}^{(3)} = 0\vert 001$
	$\mathbf{x}^{(2)} = 0\vert 001$	$\mathbf{x}^{(4)} = 0\vert 010$
2	$\mathbf{x}^{(1)} = 00\vert 10$	$\mathbf{x}^{(5)} = 00\vert 01$
	$\mathbf{x}^{(2)} = 00\vert 01$	$\mathbf{x}^{(6)} = 00\vert 10$
3	$\mathbf{x}^{(1)} = 001\vert 0$	$\mathbf{x}^{(7)} = 001\vert 1$
	$\mathbf{x}^{(2)} = 000\vert 1$	$\mathbf{x}^{(8)} = 000\vert 0$

c.) *Determine whether all your solutions in part (b) are feasible, and, if not, explain what difficulty this presents for effective application of genetic algorithm search.*

We see that solution $\mathbf{x}^{(7)} = (0,0,1,1)$ is infeasible as it violates the second constraint. All other solutions are feasible. This is easily evidenced by the fact that solutions 3-6 are simply solutions 1 and 2 renumbered. Solution 8 is also feasible because it doesn't violate any of the constraints. Since solution 7 is infeasible, we know that we will either exclude it from the population or include it with a large enough, negative objective value to limit its selection/interaction with the rest of the population.

12.55

Return again to the model of Exercise 12-45 and consider employing genetic Algorithm 12F with initial population $[(0,0,1,0),(0,0,0,1),(0,1,1,0),(1,0,0,0)]$, $p_e = 1$ and $p_c = 2$. Construct and evaluate each member of the next generation population, with crossover after component 2 of the best and worst current solutions. Use a large negative M as the objective value of any infeasible solutions produced by crossover.

Table 2: Starting Solution Objective Values

#	Solution	Objective Value	Rank
1	0010	9	3
2	0001	8	4
3	0110	16	1
4	1000	12	2

We see that starting solution 3 has the best objective value and starting solution 2 has the worst objective value. Therefore, we will select these two solutions to be our parent solutions for the genetic Algorithm crossovers.

Table 3: Genetic Algorithm Crossovers

Crossover #	Parents	Children
1	$\mathbf{x}^{(1)}01\vert 10 =$	$\mathbf{x}^{(3)} = 01\vert 01$
	$\mathbf{x}^{(2)}00\vert 01 =$	$\mathbf{x}^{(4)} = 00\vert 10$

Now, we can create the second generation population. Note, we are instructed to select only one elite and two children. Therefore, we will also include a feasible immigrant solution that is not in either of our existing solution spaces. For simplicity, solution 0000 has been chosen.

Table 4: Second Generation Objective Values

Type	Solution	Objective Value	Rank
$p_c^{(1)}$	0010	9	3
$p_c^{(2)}$	0101	15	2
$p_e^{(1)}$	0110	16	1
$p_i^{(1)}$	0000	0	4

We have therefore constructed the second generation population of the genetic Algorithm and have evaluated each solution for objective value. We see that the incumbent objective value is still 16 with corresponding solution 0110.

12.57

Consider solving (approximately) the following knapsack problem by constructive search Algorithm 12G.

$$\begin{aligned} \max\ & 11x_1 + 1x_2 + 9x_3 + 17x_4 \\ \text{s.t.}\ & 9x_1 + 2x_2 + 7x_3 + 13x_4 \leq 17 \\ & x_1, \ldots, x_4 = 0 \text{ or } 1 \end{aligned}$$

a.) *Determine the global optimum by inspection.*

We see by simple inspection that the global optimum is $\mathbf{x} = (1, 0, 1, 0)$ with an optimal objective function value of 20.

b.) *Explain why it is reasonable to fix the variables in order of ratio*

$$\frac{\text{objective coefficient}}{\text{constraint coefficient}}$$

We know that it is reasonable to fix the variables in such a ratio order since it measures effect of increase per unit of the variable by comparing the weights of the objective value function and the constraint. This is particularly true since there is only one constraint. If there were multiple constraint, this may not be the most appropriate order of ratio to use.

c.) *Apply constructive search Algorithm 12G to construct an approximate solution choosing variables to fix in this ratio sequence.*

We know that by Algorithm 12G, we want to first initialize with all-free initial partial solution $\mathbf{x}^{(0)} = (\#, \#, \#, \#)$ and a solution set index of $t \leftarrow 0$. Since none of the components of $\mathbf{x}^{(0)}$ are currently fixed, we will calculate the object of ratio values for each variable and select the 'best' (most positive) value to fix.

Variable	Object of Ratio
x_1	$\frac{11}{9} = 1.2\bar{2}2$
x_2	$\frac{1}{2} = .500$
x_3	$\frac{9}{7} = 1.286$
x_4	$\frac{17}{13} = 1.308$

Since x_4 has the best object of ratio, we will select it to be fixed and fix it to 1. Now we have the following partial solution

$$\mathbf{x}^{(1)} = (\#, \#, \#, 1)$$

The solution set index is then incremented to $t \leftarrow t + 1 \implies t = 1$. The next best object of ratio is x_3 with a ratio value 1.286. Upon inspection, we see that if we were to fix $x_3 = 1$, this would violate the main constraint. Therefore, we will fix x_3 to 0 and arrive at the following partial solution

$$\mathbf{x}^{(2)} = (\#, \#, 0, 1)$$

and advance t to $t = 2$. Now, the next best object of ratio is x_1 with a ratio value of $1.2\bar{2}2$. Again, we see upon inspection that x_1 cannot be fixed to 1 as it would violate the main constraint. Therefore, we will set it to 0 and arrive at the following partial solution

$$\mathbf{x}^{(3)} = (0, \#, 0, 1)$$

by advancing t to 3. Now, we have only the one variable left to fix, x_2. We see that it is able to be fixed to $x_2 = 1$ without violating any constraints. Therefore, we will arrive at the following solution:

$$\mathbf{x}^{(4)} = (0, 1, 0, 1)$$

and advance t to 4. This has an optimal objective value of 18. All variables are now fixed and we have arrived at a local optimum.

Note, this is less than the global optimum found in part a, but it is very close.