Advanced Calculus III
Unit 6.3: Problems 1, 2b, 3, 5a, 5b, 9, 11f, 12, 13, 16

Megan Bryant

September 16, 2013

6.3.1

Let $f \in R[a,b]$. For $x \in [a,b]$, set $F(x) = \int_a^x f$. Prove that F is continuous on $[a,b]$.

Since f is bounded, $|f(x) \leq M$ for all $x \in [a,b]$.

If $x, y \in [a,b]$ with $x<y$ then $|F(y) - F(x)| = |\int_x^y f| \leq \int_x^y |f| \leq M|y-x|$.

Therefore F is Lipchitz continuous on $[a,b]$ and is also uniformily continuous.

6.2b

For $x \in [0,1]$, find $F(x) = \int_0^x f(t)dt$ for each of the following gunctions f defined on $[0,1]$. In each case verify that F is continuous on $[0,1]$, and that $F'(x) = f(x)$ at all points where f is continuous.

$F_1(x) = x$ for $0 \leq x \leq \frac{1}{2}$.

For all $x, y \in [0, \frac{1}{2})$, for all $\epsilon>0$, choose $\delta = \epsilon>0$. Then for all $|x-y|<\delta$, $|F_1(x) - F_1(y)| = |x-y|<\delta = \epsilon$. Thus, $F_1(x) = x$ is uniformly continuous on $[0, \frac{1}{2})$.

$F_1'(x) = 1 = f(x)$ for all $x \in [0, \frac{1}{2}]$.

$F_2(x) = \frac{3}{2} - 2x$ for $x \; in[0, \frac{1}{2})$.

For all $x, y \in [\frac{1}{2}, 1]$, for all $\epsilon>0$, choose $\delta = \epsilon>0$. Then, for all $|x-y|<\delta$, $|F_2(x) - F_2(y)| = |\frac{3}{2} - 2x\frac{3}{2} + 2y| = |-2(x-y)|<\delta = \epsilon$.Thus $F_2 = \frac{3}{2} - 2x$ is uniformly continuous on $[\frac{1}{2}, 1]$.

$F_2'(x) = -2 = f(x)$ for all $x \in [\frac{1}{2}, 1]$.

6.3.3a

Let $f(t)$ be defined by

$$f(t) = \begin{cases} t & 0 \leq t < 1 \\ b - t^2 & 1 \leq t \leq 2 \end{cases}$$

and let $F(x)$ be defined by $F(x) = \int_0^x f(t)dt$, $0 \leq x \leq 2$ Find $F(x)$.

$$f(t) = \begin{cases} t & 0 \leq t < 1 \\ b - t^2 & 1 \leq t \leq 2 \end{cases}$$

By straightforward computation we see that

$$F(x) = \begin{cases} \frac{x^2}{2} & 0 \leq x < 1 \\ bx - \frac{x^3}{3} & 1 \leq x \leq 2 \end{cases}$$

6.3.3b

Let $f(t)$ be defined by

$$f(t) = \begin{cases} t & 0 \leq t < 1 \\ b - t^2 & 1 \leq t \leq 2 \end{cases}$$

and let $F(x)$ be defined by $F(x) = \int_0^x f(t)dt$, $0 \leq x \leq 2$ For what value of b in the definition of f is $F(x)$ differentiable for all $x \in [0, 2]$.

From above, we know that

$$F(x) = \begin{cases} \frac{x^2}{2} & 0 \leq x < 1 \\ bx - \frac{x^3}{3} & 1 \leq x \leq 2 \end{cases}$$

$lim_{x \to 1^-} F(1) = lim_{x \to 1^-} \frac{1^2}{2} = \frac{1}{2}$, from the left.

$F(1) = b * 1 - \frac{1^3}{3} = b - \frac{1}{3}$, from the right.

Set the two sides equal to eachother. $b - \frac{1}{3} = \frac{1}{2}$

$b = \frac{2}{6} + \frac{3}{6} = \frac{5}{6}$

Therefore for $b = \frac{5}{6}$ $F(x)$ is differentiable for all $x \in [0, 2]$.

6.3.5a

Find $F'(x)$ where F is defined on $[0,1]$ as follows: $F(x) = \int_0^x \frac{1}{1+t^2} dt$.

$F(x) = \int_0^x \frac{1}{1+t^2} dt = tan^{-1}(x) - tan^{-1}(0) = tan^{-1}(x) - 0 = tan^{-1}(x)$

Therefore, $F'(x) = \frac{1}{1+x^2}$.

6.3.5b

Find $F'(x)$ where F is defined on $[0,1]$ as follows: $F(x) = \int_0^x cost^2 dt$.

$F(x) = \int_0^x cost^2 dt = \frac{1}{2}(x + sin(x)cos(x))|_0^1$

$= \frac{1}{2}(x + sin(x)cos(x)) - \frac{1}{2}(0 + sin(0)cos(0)) = \frac{1}{2}(x + sin(x)cos(x))$

Therefore, $F'(x) = cos^2(x)$

6.3.9

Let f be a continuous real-valued function on $[a,b]$,$g \in R[a,b]$ with $g(x) \geq 0$ for all $x \in [a,b]$. Prove that there exists $c \in [a,b]$ such that $\int_a^b f(x)g(x)dx = f(c) \int_a^b g(x)dx$.

Let $m = min\{f\}$ and $M = max\{f\}$ for f on $[a,b]$.

$g(x) \geq 0$ for all x.

Therefore, $m * g(x) \leq f(x)g(x) \leq M * g(x)$. (1)

For $\int_a^b g$>0, we know from (1) that $m \leq \frac{\int_a^b fg}{\int_a^b g} \leq M$.

When $\int_a^b g = 0$, we know $|\int_a^b g| = c \int_a^b g = \int_a^b cg \leq \int_a^b |f|$, for any $c \in [a,b]$.

Thus, when $\int_a^b g = 0$, $\int_a^b fg = 0$.

Therefore, there exists $c \in [a,b]$ such that $\int_a^b f(x)g(x)dx = f(c) \int_a^b g(x)dx$.

6.3.11f

Evaluate the following integral. Justify each step. $\int_1^4 \frac{1}{x\sqrt{x+1}} dx$.

$\int_1^4 \frac{1}{x\sqrt{x+1}} dx = \int \frac{1}{(\sqrt{x+1}-1)(\sqrt{x+1}+1)\sqrt{x+1}}$

Let $u = \sqrt{x+1}$ and $du = \frac{1}{2\sqrt{x+1}}$.

$= 2\int \frac{1}{(u-1)(u+1)}du = 1\int \frac{1}{2(u-1)} - \frac{1}{2(u+1)}du$

$= \ln(|u-1|) - \ln(|u+1|) = \ln(|\sqrt{x+1}-1|) - \ln(|\sqrt{x+1}+1|)$

Which implies $\int_1^4 \frac{1}{x\sqrt{x+1}} = (\ln(|\sqrt{x+1}-1|) - \ln(|\sqrt{x+1}+1|))|_1^4$

$= \ln(\sqrt{5}-1) - \ln(\sqrt{5}+1) + \ln(\sqrt{2}+1) - \ln(\sqrt{2}-1)$.

6.3.12

Suppose f is continuous on $[0,1]$. Prove that $lim_{n\to\infty}\int_0^1 f(x^n)dx = f(0)$.

$\lim_{n\to\infty}\int_0^1 f(x^n)dx = \lim_{n\to\infty} f(x^n)|_0^1 - \int_0^1 f'(x^n)nx^n dx$

$= \lim_{n\to\infty}[f(1) - 0 - \int_0^1 f'(u) * u^{\frac{1}{n}}du]$, where $u = x^n$ and $du = nx^{n-1}dx$.

$= f(1) - \int_0^1 f'(u)(\lim_{n\to\infty} u^{\frac{1}{n}})du$

$= f(1) - \int_0^1 f'(u) * 1du = f(1) - (f(1) - f(0)) = f(0)$.

Thus, $lim_{n\to\infty}\int_0^1 f(x^n)dx = f(0)$.

6.3.13

Suppose $f : [a,b] \to R$ is continuous. Let $M = max\{|f(x)| : x \in [a,b]\}$. Show that $lim_{n\to\infty}(\int_a^b |f(x)|^n dx)^{\frac{1}{n}} = M$.

Since f is continuous, for all $\epsilon>0$, there exists δ such that $f(x)>M-\epsilon$ for all $x \in (x_0-\delta, x_0+\delta)$.

$\int_a^b f(x)^n dx > \int_{x_0-\delta}^{x_0+\delta} f(x)^n dx > \int_{x_0-\delta}^{x_0+\delta}(M-\epsilon)^n dx = (M-\epsilon)^n 2\delta$

Therefore, for any $\epsilon>0$, $(\int_a^b f(x)^n dx)^{\frac{1}{n}} > ((M-\epsilon)^n 2\delta)^{\frac{1}{n}}$

$lim_{n\to\infty}(\int_a^b f(x)^n)^{\frac{1}{n}} \geq lim_{n\to\infty}(M-\epsilon)2\delta^{\frac{1}{n}} = (M-\epsilon)$.

However, since ϵ is arbitrary, we can conclude that $lim_{n\to\infty}(\int_a^b |f(x)|^n dx)^{\frac{1}{n}} = M$.

6.3.16

Cauchy-Schawrz Inequality for Integrals: Let $f, g \in R[a, b]$. *Prove that* $|\int_a^b f(x)g(x)dx|^2 \leq (\int_a^b f^2(x)dx)(\int_a^b g^2(x)dx)$.

$0 \leq \int_a^b (f(x) - \alpha g(x))^2 dx$

$= \int_a^b f^2(x)dx - 2\alpha \int_a^b f(x)g(x)dx + \alpha^2 \int_a^b g^2(x)dx$

Let $\alpha = \frac{\int_a^b f(x)g(x)dx}{\int_a^b g^2(x)dx}$, which is a constant.

This implies $0 \leq \int_a^b f^2(x)dx - 2\frac{(\int_a^b f(x)g(x)dx)^2}{\int_a^b g^2(x)dx} + \frac{(\int_a^b f(x)g(x)dx)^2(\int_a^b g^2(x)(dx))}{(\int_a^b g^2(x)dx)^2}$

$= \int_a^b f^2(x)dx - \frac{(\int_a^b f(x)g(x)dx)^2}{\int_a^b g^2(x)dx}$.

This implies that $\frac{(\int_a^b f(x)g(x)dx)^2}{\int_a^b g^2(x)dx} \leq \int_a^b f^2(x)dx$.

Thus, $|\int_a^b f(x)g(x)dx|^2 \leq (\int_a^b f^2(x)dx)(\int_a^b g^2(x)dx)$