Advanced Calculus II
Unit 6.4: Exercises: 6.4.1a, 6.4.1b, 6.4.2f, 6.4.2g, 6.4.6, 6.4.8
Unit 6.5: Exercises 6.5.1, 6.5.3

Megan Bryant

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6.4.1a

For the following function $f(x) = \frac{1}{x^p}, 0<p<1$ *defined on* $(0,1)$*, determine whether the the improper integral of* f *converges. If it converges, find* $\int_0^1 f$*.*

For $p \in (0,1)$, $\int_0^c x^{-p} dx = \frac{c^{1-p}}{1-p}$.

$\lim\limits_{c\to 0^+} \int_c^b \frac{1}{x^p} = \lim\limits_{c\to 0^+} \frac{b^{1-p}}{p-1} - \lim\limits_{c\to 0^+} \frac{c^{1-p}}{1-p} = \frac{b^{1-p}}{p-1} - 0 = \frac{b^{1-p}}{p-1}$

Therefore, for $0<p<1$, the improper integral $\int_0^\infty f = \lim\limits_{c\to\infty} \int_a^c \frac{1}{x^p}$ exists and it converges.

$\int_0^1 x^{-p} dx = \lim\limits_{c\to 0^+} \int_c^1 x^{-p}$,by the definition of improper Riemann integral.

$= \lim\limits_{c\to 0^+} \frac{-x^{1-p}}{p-1}|_c^1 = \lim\limits_{c\to 0^+} \frac{1}{1-p}(1-c^{1-p}) = \frac{1}{1-p}$

6.4.1b

For the following function $f(x) = \frac{x}{\sqrt{1-x}}$ *defined on* $(0,1)$*, determine whether the the improper integral of* f *converges. If it converges, find* $\int_0^1 f$*.*

$\int_a^c \frac{x}{\sqrt{1-x}} = (\frac{-2}{3}x - \frac{4}{3})\sqrt{1-x}$

$\lim\limits_{c\to\infty} \int_a^c \frac{x}{\sqrt{1-x}} = \lim\limits_{c\to\infty} (\frac{-2}{3}x - \frac{4}{3})\sqrt{1-x}|_a^c$

$= \lim_{c\to\infty}(\frac{2}{3}(a+2)(\sqrt{1-a})-(\frac{2}{3}c)(\sqrt{1-c})-(\frac{4}{3})(\sqrt{1-c}))$, which does not exist.

Since the $\lim_{c\to\infty}\int_a^c f$ does not exit, the improper integral diverges.

6.4.2f

Determne whether the improper integral converges or diverges. If it converges, evaluate the integral. $\int_2^\infty \frac{dx}{x(\ln x)^p}, p>1.$

Since $p>1$, we know that the$\int_a^c f = \int_a^c \frac{dx}{x(\ln x)^p} = \frac{(\ln x)^{1-p}}{1-p}$

$\lim_{c\to\infty}\int_a^c f = \lim_{c\to\infty}\frac{(\ln x)^{1-p}}{1-p}|_a^c = \lim_{c\to\infty}\frac{(\ln a)^{1-p}}{1-p} - \frac{(\ln c)^{1-p}}{1-p}$
$= \frac{(\ln a)^{1-p}}{1-p} - \lim_{c\to\infty}\frac{(\ln c)^{1-p}}{1-p} = \frac{(\ln a)^{1-p}}{1-p}$

Since the $\lim_{c\to\infty}\int_a^c f = \frac{(\ln a)^{1-p}}{1-p}$, the improper integral converges.

Therefore $\int_2^\infty \frac{dx}{x(\ln x)^p} = \lim_{c\to\infty}\int_2^c \frac{dx}{x(\ln x)^p} = \frac{(\ln 2)^{1-p}}{1-p}$,by the definition of improper Riemann integral if $p>1$.

6.4.2g

Determine whether the improper integral converges or diverges. If it converges, evaluate the integral. $\int_0^\infty \frac{x}{x^2+1}dx.$

$\int_a^c f = \int_a^c \frac{x}{x^2+1}dx = \frac{\ln(x^2+1)}{2|_a^c} = \frac{1}{2}\ln(\frac{c^2+1}{a^2+1})$

$\lim_{c\to\infty}\int_a^c f = \lim_{c\to\infty}\frac{1}{2}\ln(\frac{c^2+1}{a^2+1}) = \infty$

Since the limit does not exist, the improper integral diverges.

6.4.6

Prove the comparison test: Let $g : [a,\infty) \to \mathbb{R}$ *be a nonnegative function satisfying* $g \in R[a,c]$ *for every* $c>a$ *and* $\int_a^\infty g(x)dx<\infty$. *If* $f : [a,\infty) \to \mathbb{R}$ *satisfies (a)* $f \in R[a,c]$ *for every* $c>a$ *and (b)*$|f(x)| \leq g(x)$ *for all* $x \in [a,\infty)$, *then the improper integral of* f *on* $[a,\infty)$ *converges and* $|\int_a^\infty f(x)dx| \leq \in_a^\infty g(x)dx$.

Since $\int_a^\infty <\infty$, the improper intergal of $g(x)$ on $[a,\infty)$ converges.

This implies that $\int_a^\infty g(x) = t$ for some finite number t.

For all $n \in \mathbb{N}$, let $a_n = \int_{n-1}^n f(x)dx$ and $b_n = \int_{n-1}^n g(x)dx$.

Thus, $t = \sum_{n=1}^{\infty} b_n$ is convergent series of non-negative terms whose sum is t.

Since $f(x) \leq g(x)$ $g(x) - f(x) \geq 0$.

Let $h(x) = g(x) - f(x) \geq 0$ for all $x \in [a, b]$.

We know from the limit definition of integrals that $\int_a^b h(x)dx \geq 0$, which implies that $\int_a^b (g(x) - f(x))dx \geq 0$.

From theorem 6.2.1, we know that
$\int_a^b (g(x) - f(x))dx = \int_a^b g(x)dx - \int_a^b f(x)dx \geq 0$.

This implies that $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.

By theorem 6.2.2, $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx = \int_a^b f(x)dx \leq \int_a^b g(x)dx$

Therefore, $|\int_a^b f(x)dx| \leq \int_a^b g(x)dx$ and $|\int_a^\infty f(x)dx| \leq \int_a^\infty g(x)dx$.

This implies that for all $n \in \mathbb{N}, |a_n| \leq b_n$.

Since b_n converges to t, for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\sum_{n=1}^{\infty} b_n < \epsilon$.

Let s_n be the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$.

Then, for all $n > m > N$, $|s_n - s_m| \leq \sum_{n=1}^{\infty} b_n < \epsilon$.

Thus, $\sum_{n=1}^{\infty} a_n$ is convergent, which implies the improper integral of f on $[a, \infty)$ converges.

6.4.8

Show that $\int_0^\infty x^{-p} \sin x dx$ *converges for all* $p \in (0, 2)$.

The Dirichlet test states:
Let $f, g : [a, \infty) \to \mathbb{R}$ *be such that:*
a.) g *is monotone decreasing and* $\lim_{x \to \infty} g(x) = 0$ *and*

b.)f is continuous and there is an M such that $|\int_a^x f(t)dt| \leq M$ *for all* $x>a$. *Then* $\int_a^\infty f(t)g(t)dt$ *converges.*

Since we haven't encountered the Dirichlet test, a proof is included:

Let $F(x) = \int_a^x f(t)dt$. Then $\lim_{b\to\infty} \int_a^b f(x)g(x)dx = \lim_{b\to\infty} \int_a^b g(x)dF(x)$

$= \lim_{b\to\infty} g(b)F(b) - g(a)F(a) - \lim_{b\to\infty} \int_a^b F(x)g'(x)dx$

$= 0 - 0 - \lim_{b\to\infty} \int_a^b F(x)g'(x)dx$.

Since $|F(x)| = |\int_a^x f(t)dt| \leq M$ and g is monotone decreasing, we know that $|\int_a^b F(x)g'(x)dx| \leq M|g(a) - g(b)|$.

Thus, $|\lim_{b\to\infty} \int_a^b f(x)g(x)dx| \leq M|g(a)|$ and $\int_a^\infty f(t)g(t)dt$ converges.

Now that the Dirichlet test is proven, we will use it to prove that $\int_0^\infty x^{-p} \sin x dx$ converges for all $p \in (0,2)$.

For all $p \in (0,2)$, let $g(x) = x^{-p}$. $g(x)$ is monotone decreasing and $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} x^{-p} = 0$.

Let $f(x) = sin(x)$. We know that $sin(x)$ is continuous and bounded. Therefore, there exists $M = 2$ such that
$|\int_0^x f(t)dt| = \int_0^x sin(x)| = |1 - cos(x)| \leq M = 2$ for all $x>0$.

Therefore $\int_a^\infty g(t)f(t)dt = \int_0^\infty x^{-p} \sin x dx$ converges for all $p \in (0,2)$.

6.5.1

Evaluate $\int_{-1}^1 f(x)d\alpha(x)$ *where f is bounded on* $[-1,1]$ *and continuous at* 0 *and* α *is given by*

$$\alpha(x) = \begin{cases} -1 & ,x<0 \\ 0 & ,x=0 \\ 1 & ,x>0 \end{cases}$$

Since $\alpha(0) = 0$, we know that $f(\alpha(0)) = f(0)$.

Since f is continuous at 0, we know that $\lim_{x\to 0} f(x) = f(0)$.

Since $\alpha(x)$ is a jump function and integrals are continuous functions, we may simplify and define

$$\alpha(x) = \begin{cases} -1 & , x<0 \\ 1 & , x \geq 0 \end{cases}$$

Let $P = \{x_0, x_1, \cdots, x_n\}$ be any partition of $[a, b]$. Since $-1<0 \leq 1$, there exists an index k, $1 \leq k \leq n$ such that $x_{k-1}<0 \leq x_k$.

Thus, $\Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1}) = 1 - (-1) = 2$ and $\Delta\alpha_i = 2$ for all $i \neq k$.

Therefore, $U(P, f, \alpha) = M_k\Delta\alpha_k = 2M_k = 2\sup\{f(t) : x_{k-1} \leq t \leq x_k\}$ and $L(P, f, \alpha) = m_k\Delta\alpha_k = 2m_k = 2\inf\{f(t) : x_{k-1} \leq x \leq x_k\}$.

Since f is continuous at 0, given $\epsilon>0$, there exists a $\delta>0$ such that $f(0) - \epsilon<f(t)<f(0) + \epsilon$ for all $t \in [-1, 1]$ with $|t - 0|<\delta$.

If P is any partition of $[a, b]$ with $x_j - x_{j-a}<\delta$ for all j, then $f(0) - \epsilon \leq 2m_k \leq 2M_k f(0) + \epsilon$.

Thus, $f(0) - \epsilon \leq 2L(P, f, \alpha) \leq 2U(P, f, \alpha) \leq f(0) + \epsilon$ and, as a conequence, $f(0) - \epsilon \leq 2 * \underline{\int_{-1}^{1}} f d\alpha \leq 2 * \overline{\int_{-1}^{1}} f d\alpha \leq f(0) + \epsilon$

Therefore, $\int_{-1}^{1} f(x)d\alpha(x) = 2f(0)$, since ϵ was arbitrary.

6.5.3a

Let α be nondecreasing on $[a, b]$. Suppose f is bounded on $[a, b]$ and integrable with respect to α on $[a, b]$. For $x \in [a, b]$, set $F(x) = \int_a^x f d\alpha$. Prove that $|F(x) - F(y)| \leq M|\alpha(x) - \alpha(y)|$ for some positive constant M and all $x, y \in [a, b]$.

Since f is bounded on $[a, b]$, $|f(x)| \leq M$ some positive constant.

We have previously demonstrated that if $f(x) \leq g(x)$ for all $x \in [a, b]$ then $\int f(x)dx \leq \int g(x)dx$.

$F(y) - F(x) = \int_a^y f d\alpha - \int_a^x f d\alpha = \int_x^y f d\alpha$, by definition 6.1.5 and theorem 6.2.3.

For $y>x$,

Since f is integrable with respect to α on $[a, b]$, we know that $\int_x^y f d\alpha \leq \int_x^y M d\alpha = M(\alpha(x) - \alpha(y))$, since $f(x) \leq M$ for all $x \in [a, b]$

and $\int_x^y f d\alpha \geq \int_x^y -M d\alpha = -M(\alpha(x) - \alpha(y))$, since $f(x) \geq -M$ for all $x \in [a,b]$.

Thus, $|F(x) - F(y)| \leq M(x - y)$.

For $x>y$,

Since f is integrable with respect to α on $[a,b]$, we know that
$\int_y^x f d\alpha \leq \int_y^x M d\alpha = M(\alpha(y) - \alpha(x))$, since $f(x) \leq M$ for all $x \in [a,b]$
and $\int_y^x f d\alpha \geq \int_y^x -M d\alpha = -M(\alpha(y) - \alpha(x))$, since $f(x) \geq -M$ for all $x \in [a,b]$.

Thus, $|F(x) - F(y)| \leq M(y - x)$.

For $x = y$, $|F(x) - F(y)| = |F(x) - F(x)| = 0 \leq M(0) = 0$.

Combining all three cases, we have $|F(x) - F(y)| \leq M|\alpha(x) - \alpha(y)|$.

6.5.3b

Let α be nondecreasing on $[a,b]$. Suppose f is bounded on $[a,b]$ and integrable with respect to α on $[a,b]$. For $x \in [a,b]$, set $F(x) = \int_a^x f d\alpha$.Prove that if α is continuous at $x_0 \in [a,b]$, then F is also continuous at x_0.

Let $x \in [a,b]$ and let $\{x_n\}_{n=0}^{\infty}$ be any sequence in $[a,b]$ converging to x.

Then we have
$-M|\alpha(x_n) - \alpha(x)| \leq F(x_n) - F(x) \leq M|\alpha(x_n) - (x)|$ for each $n \in \mathbb{N}$

Which implies $-M|x_n - x| \leq f(x_n) - f(x) \leq M|x_n - x|$ for each $n \in \mathbb{N}$ since $\alpha(x)$ is continuous and $F(x) = \int_a^x f d\alpha$.

Since both $-M|x_n - x|$ an $M|x_n - x|$ converge to 0 as $n \to \infty$, by the squeeze theorem $F(x_n) - F(x)$ converges to 0 as $n \to \infty$.

This implies that $\lim_{n\to\infty} F(x_n) = F(x)$ for all sequences $\{x_n\}_{n=0}^{\infty} \in [a,b]$ converging to x. Therefore, F is continuous at x.

Since x was chosen arbitrarily, F is continuous atx_0.