Advanced Calculus II
Unit 7.1: Exercises: 7.1.1, 7.1.2b, 7.1.2f, 7.1.2k, 7.1.2o, 7.1.3c, 7.1.4a, 7.1.4e

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7.1.1

If a and b are positive real numbers, prove that $\sum\limits_{k=1}^{\infty} \frac{1}{(ak+b)^p}$ converges if $p>1$ and diverges if $p \leq 1$.

There are three cases: $p>1, p=1, p<1$. We will evaluate each case seperately. For $p>1$, we will use the comparison test:

$a_k = \frac{1}{(ak+b)^p} \leq b_k = \frac{1}{k^p}$ for all $a, b \in \mathbb{R}$ and $p>1$ for all $k \geq k_0$ for some $k_0 \in \mathbb{N}$.

b_k is a p-series with $p>1$, thus $\sum b_k<\infty$.

Since $\sum b_k<\infty$, $\sum a_k<\infty$ by the comparison test. Thus, for $p>1$, a_k converges.

For $p=1$, we will use the integral test:

$a_k = \frac{1}{ak+b}$ is a decreasing sequence of nonnegative real numbers since $a_k \geq a_k+1$ for all $k \in \ltimes$.

Let $f(x) = \frac{1}{ax+b}$. $f'(x) = \frac{-a}{(ax+b)^2}$. Therefore $f(x)$ is a nonnegative, monotone decreasing function on $[1, \infty)$ satisfying $f(k) = a_k$ for all $k \in \mathbb{N}$.

$$\int\limits_1^{\infty} f(x) = \lim_{c\to\infty} \int\limits_1^{c} f(x)dx = \lim_{c\to\infty} \int\limits_1^{c} \frac{1}{ax+b}dx = \lim_{c\to\infty} \frac{\ln(\frac{ac+b}{a+b})}{a} = \infty$$

Therefore, by the integal test, since $\int\limits_1^{\infty} f(x) = \infty$, $\sum a_k$ diverges.

For $p<1$, we will use the comparison test:

$a_k = \frac{1}{(ak+b)^p} \leq b_k = \frac{1}{k^p}$ for all $a, b \in \mathbb{R}$ and $p<1$ for all $k \geq k_0$ for some $k_0 \in \mathbb{N}$.

b_k is a p-series with $p<1$, thus $\sum b_k > \infty$.

Since $\sum b_k > \infty$, $\sum a_k > \infty$ by the comparison test. Thus, for $p<1$, a_k diverges.

7.1.2b

Test the following series for convergence: $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^2+2k-1}$.

We will apply the comparison test:

$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^2+2k-1}$ and $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^{\frac{3}{2}}}$

$a_k = \frac{\sqrt{k}}{k^2+2k-1} \leq \frac{\sqrt{k}}{k^2} = b_k = \frac{1}{k^{\frac{3}{2}}}$ for all $k \geq 1$.

We know that since $\sum_{k=1}^{\infty} \frac{1}{k^{\frac{3}{2}}}$ is a p series with $p = \frac{3}{2}$. Therefore, it is convergent.

Therefore, the comparison test tells us that $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^2+2k-1}$ also converges.

7.1.2f

Test the following series for convergence: $\sum_{k=1}^{\infty} \frac{3^k}{k!}$.

We will apply the ratio test:

$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{3^{k+1}}{k+1} \frac{k!}{3^k} = \lim_{k\to\infty} \frac{3}{k+1} = 0.$

Therefore, the series converges.

7.1.2k

Test the following series for convergence: $\sum_{k=2}^{\infty} \frac{1}{(\ln k)^2}$.

We will apply the comparison test:

$\sum_{k=2}^{\infty} b_k = \sum_{k=2}^{\infty} \frac{1}{(\ln(k))^2}$ and $\sum_{k=2}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k}$

We know that since $\sum_{k=1}^{\infty} \frac{1}{k}$ is the harmonic series. Therefore, it is divergent.

Since $k \geq \ln(k)^2$, we know $b_k = \frac{1}{(\ln(k))^2} \geq a_k = \frac{1}{k}$ for all $k \geq \frac{1}{2}$.

Therefore, since $b_k \geq a_k$ for $k \geq \frac{1}{5}$ and $\sum_{k=2}^{\infty} b_k$ is divergent, the comparison test tells us that $\sum_{k=2}^{\infty} b_k = \sum_{k=2}^{\infty} \frac{1}{(\ln(x))^2}$ also diverges.

7.1.2o

Test the following series for convergence: $\sum_{k=1}^{\infty} \frac{1}{k} \ln(1 + \frac{1}{k})$.

We will apply the limit comparison test:

$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k} \ln(1 + \frac{1}{k})$ and $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$

We know that since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a p series with $p = 2$. Therefore, it is convergent.

$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\frac{1}{k}\ln(1+\frac{1}{k})}{\frac{1}{k^2}} = \lim_{k\to\infty} k \ \ln(\frac{1}{k} + 1) = 1$

Therefore, since the limit is 1 and $\sum_{k=1}^{\infty} b_k$ is covergent, the limit comparison test tells us that $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k} \ln(1 + \frac{1}{k})$ also converges.

7.1.3c

For the following series, determine all values of $p \in \mathbb{R}$ for which the given geometric series converges, and find the sum of the series:
$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} (\frac{1+p}{1-p})^k$ *where* $p \neq 1$.

We will use the root test:

$\alpha = \limsup_{k\to\infty} \sqrt[k]{\frac{1+p}{1-p}^k} = \limsup_{k\to\infty} \frac{1+p}{1-p} = \frac{1+p}{1-p}$

For the series to converge, α<1. Therefore $\frac{1+p}{1-p}$<1 when a_k converges.

$\frac{1+p}{1-p}<1$

$1+p<1-p$

$p<0$.

Thus, the geometric series $\sum a_k = \sum_{k=0}^{\infty} (\frac{1+p}{1-p})^k$ converges for all $p<0$.

7.1.4a

Suppose $a_k \geq 0$ for all $k \in \mathbb{N}$ and $\sum a_k<\infty$. For the following series, either prove that the given series converges, or provide an example for which the series diverges: $\sum_{k=1}^{\infty} \frac{a_k}{1+a_k}$.

We will use the comparison test:

Since $\sum a_k<\infty$ implies $\sum a_k$ converges , we know that $0 \leq \frac{a_k}{a_k+1} \leq a_k$.

Thus, by the comparison test, $\sum_{k=1}^{\infty} \frac{a_k}{1+a_k}$ converges if $\sum a_k<\infty$.

7.1.4e

Suppose $a_k \geq 0$ for all $k \in \mathbb{N}$ and $\sum a_k<\infty$. For the following series, either prove that the given series converges, or provide an example for which the series diverges: $\sum_{k=1}^{\infty} \sqrt[k]{k} a_k$.

We will use the limit comparison test:

Let $b_n = \sqrt[k]{k} a_k$ and $a_n = a_k$.

$\lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{k\to\infty} \frac{\sqrt[k]{k} a_k}{a_k} = \lim_{k\to\infty} \sqrt[k]{k} = 1.$

Therefore, by the limit comparison test, since $\sum a_n = \sum a_k$ converges, $\sum b_n = \sum \sqrt[k]{k} a_k$ converges.