DISCRETE OPTIMIZATION

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Executive Summary

Homework 6. Due April 17th, 2015.

Note: Please do eith $1(g)$ or $3(b)$ but not both.

1

Consider the following mixed $0-1$ integer linear program. It is called the uncapacitated warehouse location problem (see Parker and Rardin Ch. 5). The f_i denote the fixed cost of building warehouse i. If $y_i = 1$ then we build the warehouse and incur the fixed cost f_i. The c_{ij} denote the transportation costs from each warehouse i to each supply point j. The x_{ij} give the amount shipped from warehouse i to supply point j. Finally, the d_j denote the demand warehouse items at each supply point i. The formulation is

$$(P) \min \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}x_{ij} + \sum_{i=1}^{n} f_i y_i \tag{1}$$

$$\text{s.t. } \sum_{i=1}^{n} x_{ij} \geq d_j, \forall j = 1, \ldots, m \tag{2}$$

$$\sum_{j=1}^{m} x_{ij} \leq \left(\sum_{j=1}^{m} d_j\right) y_i, \forall i = 1, \ldots, n \tag{3}$$

$$d_j \geq x_{ij} \geq 0, \forall i = 1, \ldots, n; j = 1, \ldots, m \tag{4}$$

$$y_i \in \{0, 1\}, \forall i = 1, \ldots, n \tag{5}$$

There are several Lagrangian relaxations available for this problem. First, let's consider relaxing set (2). Then, after a bit of rearranging the Lagrangian problem is

$$(P_u) \min \sum_{j=1}^{m} u_j d_j \sum_{i=1}^{n}(f_i y_i + \sum_{j=1}^{m}(c_{ij} - u_j)x_{ij})$$
$$\text{s.t. } (2), (3), and (4)$$

P_u is separable in i yielding

$$v(P_u) = \sum_{j=1}^{m} d_j u_j + \sum_{i=1}^{n} v\Big(f_i y_i + \sum_{j=1}^{m} (c_{ij} - u_j) x_{ij}$$
$$\text{s.t } \sum_{j=1}^{m} x_{ij} \leq \left(\sum_{j=1}^{m} d_j \right) y_i$$
$$d_j \geq x_{ij} \geq 0 \forall j = 1, \ldots, m$$
$$y \in \{0,1\} \Big)$$

Note that the optimal solution to each of the i subproblems is easy to solve. If $(c_{ij} - u_j) < 0$ then $x_{ij} = d_j$ and $y_i = 1$ while if $(c_{ij} - uj) >= 0$ then $x_{ij} = 0$ and $y_i = 0$.

Consider the following small example of an uncapacitated warehouse location problem.

$$\begin{aligned}(P) \text{ min } & 8x_{11} + 7x_{12} + 5x_{21} + 4x_{22} + x_{31} + 3x_{32} + 36y_1 + 12y_2 + 36y_3 \\ \text{s.t } & x_{11} + x_{21} + x_{31} \geq 6 \\ & x_{12} + x_{22} + x_{32} \geq 6 \\ & x_{11} + x_{12} \leq 12y_1 \\ & x_{21} + x_{22} \leq 12y_2 \\ & x_{31} + x_{32} \leq 12y_3 \\ & 6 \geq x_{ij} \geq 0 \forall i = 1, \ldots, 3, \forall j = 1,2 \\ & y \in \{0,1\} \forall i = 1, \ldots, 3\end{aligned}$$

a.) *Solve the LP relaxation of P.* The linear relaxation is as follows.

$$\begin{aligned}(\bar{P}) \text{ min } & 8x_{11} + 7x_{12} + 5x_{21} + 4x_{22} + x_{31} + 3x_{32} + 36y_1 + 12y_2 + 36y_3 \\ \text{s.t } & x_{11} + x_{21} + x_{31} \geq 6 \\ & x_{12} + x_{22} + x_{32} \geq 6 \\ & x_{11} + x_{12} \leq 12y_1 \\ & x_{21} + x_{22} \leq 12y_2 \\ & x_{31} + x_{32} \leq 12y_3 \\ & 6 \geq x_{ij} \geq 0, \forall i = 1, \ldots, 3, \forall j = 1,2 \\ & 6 \geq y \geq 0, \forall i = 1, \ldots, 3\end{aligned}$$

This model was then formulated and solved in AMPL. The resulting optimal solution was $x_{22} = x_{31} = 6, x_{11} = x_{12} = x_{21} = x_{32} = 0, y_1 = 0$, and $y_2 = y_3 = \frac{1}{2}$ with an optimal objective value of 54.

b.) *Solve P_u with $u_1 = 4$ and $u_2 = 6$ by using the preceding shortcut.*

There are three subproblems that must be solved, one for each i. The structures of the subproblems is as follows.

$$
\begin{aligned}
&i = 1 \\
&\min\ 36y_1 + (8-4)x_{11} + (7-6)x_{12} \\
&\text{s.t. } x_{11} + x_{12} \le (6+6)y_1 \\
&\quad 6 \ge x_{11} \ge 0, 6 \ge x_{12} \ge 0 \\
&\quad y_1 = \{0,1\}
\end{aligned}
$$

$$
\begin{aligned}
&i = 2 \\
&\min\ 12y_2 + (5-4)x_{21} + (4-6)x_{22} \\
&\text{s.t. } x_{21} + x_{22} \le (6+6)y_2 \\
&\quad 6 \ge x_{21} \ge 0, 6 \ge x_{22} \ge 0 \\
&\quad y_2 = \{0,1\}
\end{aligned}
$$

$$
\begin{aligned}
&i = 3 \\
&\min\ 36y_3 + (1-4)x_{31} + (3-6)x_{32} \\
&\text{s.t. } x_{31} + x_{32} \le (6+6)y_3 \\
&\quad 6 \ge x_{31} \ge 0, 6 \ge x_{32} \ge 0 \\
&\quad y_3 = \{0,1\}
\end{aligned}
$$

$i = 1$		
x_{11}: x_{12}:	$c_{11} - u_1 = 8 - 4 = 4 \not< 0$ $c_{12} - u_2 = 7 - 6 = 1 \not< 0$	$\implies x_{11} = y_1 = 0$ $\implies x_{12} = y_1 = 0$
$i = 2$		
x_{21}: x_{22}:	$c_{21} - u_1 = 5 - 4 = 1 \not< 0$ $c_{22} - u_2 = 4 - 6 = -2 < 0$	$\implies x_{21} = y_2 = 0$ $\implies x_{22} = 6, y_2 = 1$
$i = 3$		
x_{32}: x_{33}:	$c_{31} - u_1 = 1 - 4 = -3 < 0$ $c_{32} - u_2 = 3 - 6 = -3 < 0$	$\implies x_{31} = 6, y_3 = 1$ $\implies x_{32} = 6, y_3 = 1$

With $i = 2$, we see that x_{21} must equal 0. However, because the objective function can then be simplified to $12y_1 - 2x_{22}$ that the cost is 0 regardless of whether or not we open facility 2 and service 2 ($y_2 = 1$ and $x_{22} = 6$. Therefore, since location 1 is closed, our dual objective function is really as follows

$$v(P_u) = x_{31} * u_1 + x_{31} * u_2 + 36y_3 - 3x_{31} - 3x_{32} = 6 * 4 + 6 * 6 + 36 - 18 - 18 = 60$$

c.) *Compare the bounds generated in* ***a.)*** *and* ***b.)****. Does P_u have the integrality property? Why or why not?*

The integrality property states: *Let $(P), (P_u)$, and (D_L) be as defined above. If T has the integrality property, then $v(D_L) = v(\bar{P})$.*

Now, we know that $v(\bar{P}) = 54$ and $v(P_u) = 60$ from parts **a.)** and **b.)** respectively. Thus, we have $v(P_u) = 60 \geq 54 = v(\bar{P})$.

We also know by definition that the $v(D_L)$ is the max of all the $v(P_u)$. Therefore, we have $v(D_L) \geq v(P_u) \geq v(\bar{P})$.

Thus, P_u can't have the integrality property.

d.) *Formulate (in general terms) a second Lagrangian problem P_u^2 from P by relaxing all type (1) constraints. Please rearrange terms so that it is in its simplest form.*

Let's first look at the explicit case. Before we begin, we need to modify the program so that the we have it in the appropriate format.

$$
\begin{aligned}
(P)\ \min\ & 8x_{11} + 7x_{12} + 5x_{21} + 4x_{22} + x_{31} + 3x_{32} + 36y_1 + 12y_2 + 36y_3 \\
\text{s.t }\ & x_{11} + x_{21} + x_{31} \geq 6 \\
& x_{12} + x_{22} + x_{32} \geq 6 \\
& 12y_1 - x_{11} - x_{12} \geq 0 \\
& 12y_2 - x_{21} - x_{22} \geq 0 \\
& 12y_3 - x_{31} - x_{32} \geq 0 \\
& 6 \geq x_{ij} \geq 0 \forall i = 1, \ldots, 3, \forall j = 1, 2 \\
& y \in \{0, 1\} \forall i = 1, \ldots, 3
\end{aligned}
$$

Now, we have the immediate Lagrangian relaxation as follows.

$$
\begin{aligned}
(P_u)\ \min\ & 8x_{11} + 7x_{12} + 5x_{21} + 4x_{22} + x_{31} + 3x_{32} + 36y_1 + 12y_2 + 36y_3 \\
& + u_1(0 - 12y_1 + x_{11} + x_{12}) + u_2(0 - 12y_2 + x_{21} + x_{22}) + u_3(0 - 12y_3 + x_{31} + x_{32}) \\
\text{s.t }\ & x_{11} + x_{21} + x_{31} \geq 6 \\
& x_{12} + x_{22} + x_{32} \geq 6 \\
& 6 \geq x_{ij} \geq 0, \forall i = 1, \ldots, 3, \forall j = 1, 2 \\
& y_i \in \{0, 1\}, \forall i = 1, \ldots, 3
\end{aligned}
$$

Now, we can simplify the objective function.

$$
\begin{aligned}
& 8x_{11} + 7x_{12} + 5x_{21} + 4x_{22} + x_{31} + 3x_{32} + 36y_1 + 12y_2 + 36y_3 \\
& - 12y_1u_1 + x_{11}u_1 + x_{12}u_1 - 12y_2u_2 + x_{21}u_2 + x_{22}u_2 - 12y_3u_3 + x_{31}u_3 + x_{32}u_3 = \\
& (8 + u_1)x_{11} + (7 + u_1)x_{12} + (5 + u_2)x_{21} + (4 + u_2)x_{22} + (1 + u_3)x_{31} + (3 + u_3)x_{32} \\
& + (36 - 12u_1)y_1 + (12 - 12u_2)y_2 + (36 - 12u_3)y_3
\end{aligned}
$$

Now, we can present the simplest form Lagrangian relation.

$$\begin{aligned}(P_u)\ \min\ & (8+u_1)x_{11}+(7+u_1)x_{12}+(5+u_2)x_{21}+(4+u_2)x_{22}+(1+u_3)x_{31}+(3+u_3)x_{32} \\ & +(36-12u_1)y_1+(12-12u_2)y_2+(36-12u_3)y_3 \\ \text{s.t}\ & x_{11}+x_{21}+x_{31}\geq 6 \\ & x_{12}+x_{22}+x_{32}\geq 6 \\ & 6\geq x_{ij}\geq 0, \forall i=1,\dots,3, \forall j=1,2 \\ & y_i\in\{0,1\}, \forall i=1,\dots,3\end{aligned}$$

Now, let's define this last simplification in general terms.

$$\begin{aligned}\min\ & \sum_j\sum_i(c_{ij}+u_i)x_{ij}+\sum_i(f_i-\sum_j d_ju_i)y_i \\ \text{s.t}\ & \sum_i x_{ij}\geq d_j, \forall j \\ & 6\geq x_{ij}\geq 0, \forall i=1,\dots,3, \forall j \\ & y_i\in\{0,1\}, \forall i=1,\dots,3\end{aligned}$$

e.) *Solve this new Lagrangian problem for the above example when $u_1=u_2=u_3=3$.*
When $u_1=u_2=u_3=3$, we have the following updated Lagrangian relaxation.

$$\begin{aligned}\min\ & \sum_j\sum_i(c_{ij}+3)x_{ij}+\sum_i(f_i-3\sum_j d_j)y_i \\ \text{s.t}\ & \sum_i x_{ij}\geq d_j, \forall j \\ & 6\geq x_{ij}\geq 0, \forall i=1,\dots,3, \forall j \\ & y_i\in\{0,1\}, \forall i=1,\dots,3\end{aligned}$$

Note that the x_{ij} and the y_{ij} are independent.

Now, since we are minimizing, we can also see that we want to make $y_i=1$ if and only if $(f_i-3\sum_j d_j)<0$ and 0 otherwise.

y_1: $36-12*3=0$
y_2: $12-12*3=-24$
y_3: $36-12*3=0$

We see that $y_1=y_2=0$ and $y_2=1$. Now, we want to minimize. Therefore, let's calculate similar values for the x_{ij} and choose them minimum number that satisfy the constraints.

x_{11}: $8+3=11$
x_{12}: $7+3=10$
x_{21}: $5+3=8$
x_{22}: $4+3=7$
x_{31}: $1+3=4$
x_{32}: $3+3=6$

We must ensure that our Lagrangian relaxation still satisfies the constraints. Therefore, we must select an i for each j such that $\sum_i x_{ij} = 6 \forall j = 1, 2$. We will select the i such that $(c_{ij} + u_{ij} = \min\{x_{ij}\}$, for all j.

Thus, we will make $x_{31} = 6$ and $x_{32} = 6$. All other x_{ij} will be set to 0 since we are minimizing.

Our solution is therefore $x_{11} = x_{12} = x_{21} = x_{22} = y_1 = y_3 = 0$, $x_{31} = x_{32} = 6$ and $y_2 = 1$. This leaves us with a $v(P_u^2) = 6 * 1 + 3 * 6 + 12 * 1 = 36$.

f.) *Explain how, given u_j for all j it is easy to solve P_u^2 (in general terms).*

We see from our general formulation of (P_u^2) that for any given u_j, $x_{ij} = 6$ if $c_{ij} + x_{ij} < 0$ and the constraints are met and $x_{ij} = 0$ otherwise. Furthermore, $y_i = 0$ if $f_i - u_i * \sum d_j \geq 0$ and $y_i = 1$ otherwise.

g.) *Does P_u^2 have the integrality property? Why or why not? This will require some thought.*

2

Perform 3 iterations of the subgradient optimization procedure for the set covering example given below. Let u be initialized as $u = (1, 1, 1)$ and start with $\pi = 2$. Use 9 as the upper bound on $v(P)$.

$$
\begin{aligned}
(P) \ \min\ & 2x_1 + 3x_2 + 4x_3 + 5x_4 \\
\text{s.t. } & x_1 + x_3 \geq 1 \\
& x_1 + x_2 + x_4 \geq 1 \\
& x_2 + x_3 + x_4 \geq 1 \\
& x_1, x_2, x_3, x_4 \in \{0, 1\}
\end{aligned}
$$

First, let's write down both the original and simplified forms of P_u.

$$
\begin{aligned}
(P_u) \ \min\ & 2x_1 + 3x_2 + 4x_3 + 5x_4 + u_1(1 - x_1 - x_3) + u_2(1 - x_2 - x_3 - x_4) + u_3(1 - x_2 - x_3 - x_4) \\
\text{s.t. } & u_i \geq 0, \forall i = 1, 2, 3
\end{aligned}
$$

This simplifes to the following.

$$
\begin{aligned}
(P_u) \ \min\ & (2 - u_1 - u_2)x_1 + (3 - u_2 - u_3)x_2 + (4 - u_1 - u_3)x_3 + (5 - u_2 - u_3)x_4 + (u_1 + u_2 + u_3) \\
\text{s.t. } & u_i \geq 0, \forall i = 1, 2, 3
\end{aligned}
$$

Now, we are ready to begin the subgradient optimization procedure.

Step 0: Initialization
$u = (1, 1, 1)$
Uperbound $= 9$
$\pi = 2$
$v^0 = -\infty$

Step 1: Solve P_u

$$\begin{aligned}
\bar{c}_1 &= 2 - u_1 - u_2 = 2 - 1 - 1 = 0 \\
\bar{c}_2 &= 3 - u_2 - u_3 = 3 - 1 - 1 = 1 \\
\bar{c}_3 &= 4 - u_1 - u_3 = 4 - 1 - 1 = 2 \\
\bar{c}_4 &= 5 - u_2 - u_3 = 5 - 1 - 1 = 3
\end{aligned}$$

Therefore, we know by the shortcut given in problem 1 that $\mathbf{x} = (0, 0, 0, 0)$ with associated $v(P_u) = 3$. Now, we can check to see if this is optimal using Theorem 5.14. Therefore, we know that if $\mathbf{x}$ is optimal, it must satisfy the following conditions

1. $r - Rx \leq 0$
2. $u * (r - Rx) = 0$

For condition 1, we have the following

$$\begin{aligned}
1 - x_1 - x_3 &= 1 - 0 - 0 = 1 \neq 0 \\
1 - x_1 - x_2 - x_4 &= 1 - 0 - 0 - 0 = 1 \neq 0 \\
1 - x_2 - x_3 - x_4 &= 1 - 0 - 0 - 0 = 1 \neq 0
\end{aligned}$$

We see that the solution isn't primal feasible so it cannot be optimal. There is thus no need to check the complementary slackness (condition 2).

Step 2: Incumbent
Recall $v^0 = -\infty$. Thus we have that $v^0 < v(P_u) = 3$. Therefore, our new incumbent, $v^1 = 3$.

Step 3: Subgradients

$$\begin{aligned}
s_1 &= 1 - x_1 - x_2 = 1 - 0 - 0 = 1 \\
s_2 &= 1 - x_1 - x_2 - x_4 = 1 - 0 - 0 - 0 = 1 \\
s_3 &= 1 - x_2 - x_3 - x_4 = 1 \\
\mathbf{s} &= (1, 1, 1)
\end{aligned}$$

Step 4: Step Size

$$\begin{aligned}
\lambda &= \pi(UB - LB) / \sum_{i=1}^{3} s_i^2 \\
&= \frac{2(9-3)}{3} \\
&= 4
\end{aligned}$$

Step 5: Update u

$$\begin{aligned} u_1^1 &= \max\{0, u_1^0 + \lambda * s_1\} \\ &= \max\{0, 1 + 4 * 1\} \\ &= 5 \\ u_2^1 &= \max\{0, u_2^0 + \lambda * s_2\} \\ &= \max\{0, 1 + 4 * 1\} \\ &= 5 \\ u_3^1 &= \max\{0, u_3^0 + \lambda * s_3\} \\ &= \max\{0, 1 + 4 * 1\} \\ &= 5 \end{aligned}$$

Thus we have $\mathbf{u}^1 = (0, 0, 0)$. Now, we can proceed to iteration 2

Step 1: Solve P_u

$$\begin{aligned} \bar{c}_1 &= 2 - u_1 - u_2 = 2 - 5 - 5 = -8 \\ \bar{c}_2 &= 3 - u_2 - u_3 = 3 - 5 - 5 = -7 \\ \bar{c}_3 &= 4 - u_1 - u_3 = 4 - 5 - 5 = -6 \\ \bar{c}_4 &= 5 - u_2 - u_3 = 5 - 5 - 5 = -5 \end{aligned}$$

Therefore, we know by the shortcut given in problem 1 that $\mathbf{x} = (1, 1, 1, 1)$ with associated $v(P_u) = -8-7-6-5+5+5+5 = -11$. Now, we can check to see if this is optimal using Theorem 5.14. Therefore, we know that if $\mathbf{x}$ is optimal, it must satisfy the following conditions

1. $r - Rx \leq 0$
2. $u * (r - Rx) = 0$

For condition 1, we have the following

$$\begin{aligned} 1 - x_1 - x_3 &= 1 - 1 - 1 = -1 \neq 0 \\ 1 - x_1 - x_2 - x_4 &= 1 - 1 - 1 - 1 = -2 \neq 0 \\ 1 - x_2 - x_3 - x_4 &= 1 - 1 - 1 - 1 = -2 \neq 0 \end{aligned}$$

We see that the solution isn't primal feasible so it cannot be optimal. There is thus no need to check the complementary slackness (condition 2).

Step 2: Incumbent

Recall $v^1 = 3$. Thus we have that $v^1 \not< v(P_u) = -11$. Thus, we keep our original incumbent and $v^2 = 3$.

Step 3: Subgradients

$$
\begin{aligned}
s_1 &= 1 - x_1 - x_2 = 1 - 1 - 1 = -1 \\
s_2 &= 1 - x_1 - x_2 - x_4 = 1 - 1 - 1 - 1 = -2 \\
s_3 &= 1 - x_2 - x_3 - x_4 = 1 - 1 - 1 - 1 = -2 \\
\mathbf{s} &= (-1, -2, -2)
\end{aligned}
$$

Step 4: Step Size

$$
\begin{aligned}
\lambda &= \pi(UB - LB) / \sum_{i=1}^{3} s_i^2 \\
&= \frac{2(9-3)}{1+4+4} \\
&= \frac{12}{9} \\
&= \frac{4}{3}
\end{aligned}
$$

Step 5: Update u

$$
\begin{aligned}
u_1^2 &= \max\{0, u_1^1 + \lambda * s_1\} \\
&= \max\{0, 5 + \frac{4}{3} * -1\} \\
&= \frac{11}{3} \\
u_2^2 &= \max\{0, u_2^1 + \lambda * s_2\} \\
&= \max\{0, 5 + \frac{4}{3} * -2\} \\
&= \frac{7}{3} \\
u_3^2 &= \max\{0, u_3^1 + \lambda * s_3\} \\
&= \max\{0, 5 + \frac{4}{3} * -2\} \\
&= \frac{7}{3}
\end{aligned}
$$

Therefore, we have $\mathbf{u}^2 = \left(\frac{11}{3}, \frac{7}{3}, \frac{7}{3}\right)$. We can now proceed to iteration 3.

Step 1: Solve P_u

$$\bar{c}_1 = 2 - u_1 - u_2 = 2 - \frac{11}{3} - \frac{7}{3} = -4$$
$$\bar{c}_2 = 3 - u_2 - u_3 = 3 - \frac{7}{3} - \frac{7}{3} = -\frac{5}{3}$$
$$\bar{c}_3 = 4 - u_1 - u_3 = 4 - \frac{11}{3} - \frac{7}{3} = -2$$
$$\bar{c}_4 = 5 - u_2 - u_3 = 5 - \frac{7}{3} - \frac{7}{3} = \frac{1}{3}$$

Therefore, we know by the shortcut given in problem 1 that $\mathbf{x} = (1, 1, 1, 0)$ with associated $v(P_u) = -4 - \frac{5}{3} - \frac{2}{3} + \frac{11}{3} + \frac{7}{3} + \frac{7}{3} = \frac{2}{3}$. Now, we can check to see if this is optimal using Theorem 5.14. Therefore, we know that if $\mathbf{x}$ is optimal, it must satisfy the following conditions

1. $r - Rx \leq 0$
2. $u * (r - Rx) = 0$

For condition 1, we have the following

$$1 - x_1 - x_3 = 1 - 1 - 1 = -1 \neq 0$$
$$1 - x_1 - x_2 - x_4 = 1 - 1 - 1 - 0 = -1 \neq 0$$
$$1 - x_2 - x_3 - x_4 = 1 - 1 - 1 - 0 = -1 \neq 0$$

We see that the solution isn't primal feasible so it cannot be optimal. There is thus no need to check the complementary slackness (condition 2).

Step 2: Incumbent
Recall $v^2 = 3$. Thus we have that $v^2 \not< v(P_u) = \frac{2}{3}$. Thus, we keep our original incumbent and $v^3 = 3$.

Step 3: Subgradients

$$s_1 = 1 - x_1 - x_2 = 1 - 1 - 1 = -1$$
$$s_2 = 1 - x_1 - x_2 - x_4 = 1 - 1 - 1 - 0 = -1$$
$$s_3 = 1 - x_2 - x_3 - x_4 = 1 - 1 - 1 - 0 = -1$$
$$\mathbf{s} = (-1, -1, -1)$$

Step 4: Step Size

$$\lambda = \pi(UB - LB) / \sum_{i=1}^{3} s_i^2$$
$$= \frac{2(9-3)}{3}$$
$$= \frac{12}{3}$$
$$= 4$$

Step 5: Update u

$$
\begin{aligned}
u_1^3 &= \max\{0, u_1^2 + \lambda * s_1\} \\
&= \max\{0, \frac{11}{3} + 4 * -1\} \\
&= -\frac{1}{3} \\
u_2^3 &= \max\{0, u_2^2 + \lambda * s_2\} \\
&= \max\{0, \frac{7}{3} + 4 * -1\} \\
&= -\frac{5}{3} \\
u_3^3 &= \max\{0, u_3^2 + \lambda * s_3\} \\
&= \max\{0, \frac{7}{3} + 4 * -1\} \\
&= -\frac{5}{3}
\end{aligned}
$$

Thus we have $\mathbf{u}^3 = (-\frac{1}{3}, -\frac{5}{3}, -\frac{5}{3})$.

3

The generalized assignment problem is

$$
\begin{aligned}
(P)\ \min\ & \sum_i \sum_j c_{ij} x_{ij} && (6) \\
\text{s.t.} & \sum_i x_{ij} \geq 1, \forall j && (7) \\
& \sum_j a_j x_{ij} \leq s_i, \forall i && (8) \\
& x_{ij} \in \{0,1\}, \forall i,j && (9)
\end{aligned}
$$

where a_j is the positive integer size of object j and s_i is the positive integer capacity of location i.

a.) *Form the Lagrangian problem that results when constraints of type (8) are relaxed. Be sure to simplify the formulation so that it is in its clearest form.*

We need to begin by rewriting the constraints of type (8) into the general form.

$$(P)\ \min \sum_i \sum_j c_{ij} x_{ij} \tag{10}$$
$$\text{s.t.} \sum_i x_{ij} \geq 1, \forall j \tag{11}$$
$$s_i - \sum_j a_j x_{ij} \geq 0, \forall i \tag{12}$$
$$x_{ij} \in \{0,1\}, \forall i,j \tag{13}$$

First, we can bring the constraints of type (8) to the objective function by weighting them with a nonnegative vector u.

$$(P)\ \min \sum_i \sum_j x_{ij} c_{ij} + \sum_i u_i (0 - s_i + \sum_j x_{ij}) \tag{14}$$
$$\text{s.t.} \sum_i x_{ij} \geq 1, \forall j \tag{15}$$
$$x_{ij} \in \{0,1\}, \forall i,j \tag{16}$$

Now, we need to begin simplifying the objective function.

$$\begin{aligned} \sum_i \sum_j c_{ij} x_{ij} + \sum_i u_i (-s_i + \sum_j x_{ij}) &= \sum_i \sum_j c_{ij} x_{ij} - \sum_i u_i s_i + \sum_i u_i \sum_j x_{ij} \\ &= \sum_j \sum_i (c_{ij} + u_i) x_{ij} - \sum_i u_i s_i \end{aligned}$$

Therefore, we have the following Lagrangian Relaxation.

$$(P)\ \min \sum_j \sum_i (c_{ij} + u_i) x_{ij} - \sum_i u_i s_i \tag{17}$$
$$\text{s.t.} \sum_i x_{ij} \geq 1, \forall j \tag{18}$$
$$x_{ij} \in \{0,1\}, \forall i,j \tag{19}$$

However, we can make one further simplification: since the decision variables are binary, we can modify (14) for the following clearest, simplified form.

$$(P)\ \min \sum_j \sum_i (c_{ij} + u_i) x_{ij} - \sum_i u_i s_i \tag{20}$$
$$\text{s.t.} \sum_i x_{ij} = 1, \forall j \tag{21}$$
$$x_{ij} \in \{0,1\}, \forall i,j \tag{22}$$

b.) *Show that this P_u problem has the integrality property.*

Integrality Property: *If the convex hull, $[T]$, is equal to the linear relaxation, $\bar{T}$, then T is said to have the integrality property. (Note, if $v(D_L) > v(\bar{P})$, then $[T] \neq \bar{T}$. This is equivalent to saying that if $v(D_L) = v(\bar{P})$, then T has the integrality property.*

Now, based on the clearest simplification of the Lagrangian relaxation, we see that $v(P_u) = v\bar{P})$ since the presence of constraint (17) in the linear relaxation enforces the binary integer constraint regardless of the linear relaxation. Therefore, for all k, $v(P_u^k) = v(\bar{P})$. Since $v(D_L) = \max\{v(P_u^k)\}$, we have $v(D_L) = v(\bar{P})$. Thus, by definition, $v(P_u)$ has the integrality property.

A

a.) Linear Relaxation AMPL Mod:

```
param n;
param m;
param c{i in 1..m, j in 1..n};
param f{i in 1..m};
param d{j in 1..n};

var x{i in 1..m, j in 1..n} >= 0, <= 6;
var y{i in 1..m} >= 0, <= 1;

minimize ObjFun:
sum{i in 1..m, j in 1..n} c[i,j]*x[i,j] + sum{i in 1..m}f[i]*y[i];

subject to Constraints{ j in 1..n}:
sum{i in 1..m}x[i,j] >= d[j];

subject to Constraint2{i in 1..m}:
sum{j in 1..n}x[i,j] <= sum{j in 1..n}d[j]*y[i];
```

Linear Relaxation AMPL Dat:

```
set Xi;

param outcome{Xi};
param p{Xi} >= 0, <= 1;

var x{i in Xi} >= 0;
var y{i in Xi};

minimize ObjFun:
sum{i in Xi}(1.0625*x[i] + p[i]*outcome[i]*y[i]);

subject to Constraint_One{i in Xi}:
y[i] >= 1-x[i];

subject to Constraint_Two{i in Xi}:
y[i] >= 1 - outcome[i]/4;
```

AMPL Output:

```
ampl: reset; model linear.mod; data linear.dat; option solver gurobi; solve;
Gurobi 6.0.0: optimal solution; objective 54
2 simplex iterations
ampl: display x,y;
x :=
1 1 0
1 2 0
2 1 0
2 2 6
3 1 6
```

```
3 2 0
;

y [*] :=
1 0
2 0.5
3 0.5
;
```