MTHSC 4420:
Advanced Mathematical Programming
Homework 1

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Knights, Knaves, and Werewolves

Suppose that you are visiting a forest in which every inhabitant is either a knight or a knave. Knights always tell the truth and Knaves always lie. In addition, some of the inhabitants are werewolves and have the annoying habit of sometimes turning into wolves at knight and devouring people. A werewolf can be either a knight or a knave.

1.a Werewolves II

You are interviewing three inhabitants, A, B, and C. It is known that exactly one of them is a werewolf. They make the following statements:

A : I am a werewolf
B : I am a werewolf
C : At most one of us is a knight

Give a complete classification of A, B, and C solution.

Let $x_i = \begin{cases} 1 & \text{if } i \text{ is a knight} \\ 0 & \text{if } i \text{ is a knave} \end{cases}$, for $i = 1, 2, 3$.

Let $y_i = \begin{cases} 1 & \text{if } i \text{ is a werewolf} \\ 0 & \text{if } i \text{ is a human} \end{cases}$, for $i = 1, 2, 3$.

Maximize x_1
subject to:
$y_1 + y_2 + y_3 = 1$ (only one werewolf)
$y_1 = x_1$ (if A is a knight, then statement A is true)
$y_2 = x_2$ (If B is a knight, then statement B is true)
$x_1 + x_2 + 3x_3 \geq 2$
$x_1 + x_2 + 4x_3 \leq 4$
$x_i, y_i \in \{0, 1\}$, for $i = 1, 2, 3$

Based on this formulation, we can see that C must be a knight and a werewolf.

AMPL Code
AMPL Model Code:

```
param person = 3; # the number of people

var x{1..person} binary;
# x(i) = 1 if person i is a knight, 0 if a knave
var y{1..person} binary;
# y(i) = 1 if person i is a werewolf, 0 otherwise

minimize any: x[1];

subject to One_Werewolf: sum{i in 1..person} y[i] = 1;
#there is only one werewolf
subject to A_Knight: y[1] - x[1] = 0;
# if A is a knight, then statement A is true
subject to B_Knight: y[2] - x[2] = 0;
# if B is a knight, then statment B is true
subject to C_Knight: x[1] + x[2] + 3 * x[3] >= 2;
# if C is a knight, then A and B must be knaves
subject to C_Knave: x[1] + x[2] + 4 * x[3] <= 4;
#if C is a knave, then either A or B is a knight
```

AMPL Solution

```
Gurobi 5.5.0: optimal solution; objective 0
:   x   y    :=
1   0   0
2   0   0
3   1   1
;
```

1.b Werewolves IV

This time, we get the following statements:

A: At least one of the three of us is a knave
B: C is a knight

Given that there is exactly one werewolf and that he is a knight, who is the werewolf?

Let $x_i = \begin{cases} 1 & \text{if } i \text{ is a knight} \\ 0 & \text{if } i \text{ is a knave} \end{cases}$, for $i = 1, 2, 3$.

Let $y_i = \begin{cases} 1 & \text{if } i \text{ is a werewolf} \\ 0 & \text{if } i \text{ is a human} \end{cases}$, for $i = 1, 2, 3$.

Maximize x_1
subject to:
$4x_1 + x_2 + x_3 \geq 3$
$4x_1 + x_2 + x_3 \leq 5$
$x_3 = x_2$
$y_1 + y_2 + y_3 = 1$
$x_1 \geq y_1$
$x_2 \geq y_2$
$x_3 \geq y_3$
$x_i, y_i \in \{0, 1\}$, for $i = 1, 2, 3$

Based on this formulation, we can see that A must be a knight and a werewolf.

AMPL Code

AMPL Model Code:

```
param person = 3;
#the number of people

var x{1..person} binary;
# x(i) = 1 if person i is a knight, 0 if a knave
var y{1..person} binary;
# y(i) = 1 if person i is a werewolf, 0 otherwise

minimize any: x[1];

subject to One_Werewolf: sum{i in 1..person} y[i] = 1;
#there is only one werewolf
subject to A_Knight: 4 * x[1] + x[2] + x[3] >= 3;
#If A is a knight, than B or C or B and C are knaves
subject to A_Knave: 4 * x[1] + x[2] + x[3] <= 5;
#If A is a knave, then no one is a knave
subject to B_C: x[2] - x[3] = 0;
# B and C are either both knights or both knaves
subject to A_A: x[1] - y[1] >= 0;
# A is either a knight and a werewolf or a knave
subject to B_B: x[2] - y[2] >= 0;
# B is either a knight and a werewolf or a knave
subject to C_C: x[3] - y[3] >= 0;
# C is either a knight and a werewolf or a knave
```

AMPL Solution

```
Gurobi 5.5.0: optimal solution; objective 1
:   x   y    :=
1   1   1
2   0   0
3   0   0
;
```

Clark County EMS Optimization

The emergency services coordinator of Clarke County is planning to purchase new EMS vehicles to serve the county's residents. The county is divided into five districts. The standard of care that she wishes to meet requires that every district be reachable by a vehicle within four minutes. The average times required to travel from one region to the next are summarized in the following table:

A =

	To					
From	1	2	3	4	5	6
1	0	4	3	6	6	5
2	4	0	7	5	5	6
3	3	7	0	4	3	5
4	6	5	4	0	7	5
5	6	5	3	7	0	2
6	5	6	5	5	2	0

The populations in regions 1 through 6 (in 1000's) are estimated to be 21, 35, 15, 60, 20, and 37 respectively.

2

The coordinator needs to decide how many vehicles to purchase and which regions to locate them in. Formulate as an integer linear program the problem of finding the minimum number of vehicles necessary and choosing their locations.

A =

*	1	2	3	4	5	6
1	0	4	3	6	6	5
2	4	0	7	5	5	6
3	3	7	0	4	3	5
4	6	5	4	0	7	5
5	6	5	3	7	0	2
6	5	6	5	5	2	0

B =

*	1	2	3	4	5	6
1	0	1	1	0	0	0
2	1	0	0	0	0	0
3	1	0	0	1	1	0
4	0	0	1	0	0	0
5	0	0	1	0	0	1
6	0	0	0	0	1	0

B is the incidence matrix of A where a 1 represents a time less than four minutes ($a_{ij} \leq 4$) and a 0 represents a time greater than four minutes.

Let $D = \{1, \cdots, 6\}$ represent the set of all possible districts.

Let b_{ij} represent a position in the incidence matrix B.

Let $x_j = \begin{cases} 1 & \text{if district } j \text{ is chosen to receive a vehicle} \\ 0 & \text{otherwise} \end{cases}$, for $j \in D$.

Minimize $\sum_{j \in D} x_j$
subject to:
$\sum_{j \in D} b_{ij} x_j \geq 1, i \in D$
$x_j \in \{0, 1\}$

Based on this formulation, we would make the following recommendation to the coordinator:

Clarke County EMS Coordinator,
Based on the specifications provided, it is recommended that three EMS vehicles be purchased and these vehicles be placed in districts 1, 3, and 5. This will ensure that every district can be serviced within four minutes.

AMPL Code
AMPL Model Code:

```
param district = 6;
param B {i in 1..district, j in 1..district};

var x{1..district} binary; #1 if region j is chosen to receive a vehicle, 0 otherwise

minimize Number_of_Vehicles: sum{j in 1..district} x[j]; #Minimizing the number of ve

subject to Min_Vehicles {i in 1..district}: sum{j in 1..district} B[i,j] * x[j] >= 1;
```

AMPL Data Code:

```
param B: 1 2 3 4 5 6 :=
1 0 1 1 0 0 0
2 1 0 0 0 0 0
3 1 0 0 1 1 0
4 0 0 1 0 0 0
5 0 0 1 0 0 1
6 0 0 0 0 1 0;
```

AMPL Run Code:

```
reset;
model problem2.mod;
data problem2.dat;
option solver gurobi;
solve;
display Number_of_Vehicles;
display x;
```

AMPL Solution

```
Number_of_Vehicles = 3

x [*] :=
1  1
2  0
3  1
4  0
5  1
6  0
;
```

3

The Clarke County Board of Supervisors has allocated a budget that is enough to purchase two EMS vehicles. The emergency services coordinator wants to determine the districts in which to locate the vehicles to maximize the number or residents in districts reachable within four minutes. Formulate this problem as an integer linear program

*	1	2	3	4	5	6
1	0	4	3	6	6	5
2	4	0	7	5	5	6
3	3	7	0	4	3	5
4	6	5	4	0	7	5
5	6	5	3	7	0	2
6	5	6	5	5	2	0

A =

*	1	2	3	4	5	6
1	0	1	1	0	0	0
2	1	0	0	0	0	0
3	1	0	0	1	1	0
4	0	0	1	0	0	0
5	0	0	1	0	0	1
6	0	0	0	0	1	0

B =

B is the incidence matrix of A where a 1 represents a time less than four minutes ($a_{ij} \leq 4$) and a 0 represents a time greater than four minutes.

Let b_{ij} represent a position in the incidence matrix B.

Let $D = \{1, \cdots, 6\}$ represent the set of all possible districts.

P = 21 35 15 60 20 37

Let $x_j = \begin{cases} 1 & \text{if district } j \text{ is chosen to receive a vehicle} \\ 0 & \text{otherwise} \end{cases}$, for $j \in D$.

Let $y_i = \begin{cases} 1 & \text{if district } i \text{ is not serviced} \\ 0 & \text{otherwise} \end{cases}$, for $i \in D$.

Minimize $\sum\limits_{i \in D} P[i]y[i]$
subject to:
$\sum_{j \in D} b_{ij}x_j + y_i \geq 1$
$\sum_{j \in D} x_j \leq 2$
$x_j \in \{0, 1\}$, for $j \in D$
$y_i \in \{0, 1\}$, for $i \in D$

Based on this formulation, we would make the following recommendation to the coordinator:

Clarke County EMS Coordinator,
Based on the specifications provided, districts 3 and 5 should recieve vehicles. This will minimize the number of districts that cannot be serviced with four minutes to district 2. All other districts will be reachable within 4 minutes, on average.

AMPL Code

AMPL Model Code:

```
param district = 6;
param B {i in 1..district, j in 1..district};
param P {i in 1..district};

var x{1..district} binary;
#1 if region j is chosen to receive a vehicle, 0 otherwise
var y{1..district} binary;
#1 if region i is not serviced, 0 otherwise
minimize Not_Serviced:sum{i in 1..district}P[i]*y[i];

#Minimize the population that is not served

subject to Service_Check  {i in 1..district}: sum{j in 1..district} B[i,j] * x[j] + y
#Verify that every district that is serviced is serviced in less than four minutes
subject to Max_Num_Vehicles: sum{j in 1..district} x[j] <=2;
#Set the maximum number of vehicles to 2
```

AMPL Data Code:

```
param B: 1 2 3 4 5 6 :=
1 0 1 1 0 0 0
2 1 0 0 0 0 0
3 1 0 0 1 1 0
4 0 0 1 0 0 0
5 0 0 1 0 0 1
6 0 0 0 0 1 0;
```

AMPL Run Code:

```
reset;
model problem3.mod;
data problem3.dat;
option solver gurobi;
solve;
display Not_Serviced;
display y;
display x;
```

AMPL Solution

```
ampl: include run.run;
Gurobi 5.5.0: optimal solution; objective 35
Not_Serviced = 35

y [*] :=
1  0
2  1
3  0
4  0
5  0
6  0
;

x [*] :=
1  0
2  0
3  1
4  0
5  1
6  0
;
```

4

The coordinator wants to locate the two vehicles to minimize the maximum amount of time it takes to reach any district. Formulate this problem as an integer linear program.

A =

*	1	2	3	4	5	6
1	0	4	3	6	6	5
2	4	0	7	5	5	6
3	3	7	0	4	3	5
4	6	5	4	0	7	5
5	6	5	3	7	0	2
6	5	6	5	5	2	0

Let $D = \{1, \cdots, 6\}$ represent the set of all possible districts.

Let $y_j = \begin{cases} 1 & \text{if district } j \text{ is chosen to receive a vehicle} \\ 0 & \text{otherwise} \end{cases}$, for $j \in D$.

Let $x_{ij} = \begin{cases} 1 & \text{if district } i \text{ is served by a vehicle in district } j \\ 0 & \text{otherwise} \end{cases}$, for $j \in D$.

Let a_{ij} represent a position in the incidence matrix A and reflect the time to serve district j from district i.

Let $T = \max\{a_{ij}\}$ for $i \in D, j \in D$. T is the maximum service time.

Minimize $\sum\limits_{i \in D} \sum\limits_{j \in D} a_{ij} x_{ij}$
subject to:
$\sum\limits_{i \in D} \sum\limits_{j \in D} a_{ij} x_{ij} - \sum_{j \in D} a_{ij} x_{ij} \geq 0$ for $i \in D$
$\sum_{j \in D} x_{ij} = 1$ for $i \in D$
$x_{ij} \leq y_j$ for $i \in D, j \in D$
$\sum_{j \in D} y_j \leq 2$
$y_j \in \{0, 1\}$, for $j \in D$
$x_{ij} \in \{0, 1\}$, for $i \in D$

Based on this formulation, we would make the following recommendation to the coordinator:

Clarke County EMS Coordinator,
If Clarke County has the budget for 2 EMS vehicles and wishes to minimize the maximum service time of each region, it is recommended that vehicles be assigned to districts 3 and 5. The vehicle assigned to district 3 should be responsible for servicing districts 1, 3, and 4. The vehicle assigned to district 5 should be responsible for servicing districts 2, 5, and 6. This will minimize the maximum service time to 7 minutes.

AMPL Code

AMPL Model Code:

```
param district = 6;
param A {i in 1..district, j in 1..district};
param T = max{i in 1..district, j in 1..district} A[i,j];
#This is the maximum service time

var y{1..district} binary;
#1 if district j is chosen to receive a vehicle, 0 otherwise
var x{1..district,1..district} binary;
#1 if district i is serviced by district j, 0 otherwise

minimize Time:
sum{i in 1..district, j in 1..district}A[i,j]*x[i,j];

subject to One_Vehicle {i in 1..district}:
sum{j in 1..district} x[i,j] = 1;
#Making sure that only one vehicle services a district
subject to Max_Districts {i in 1..district, j in 1..district}:
x[i,j] <= y[j];
#Setting Max number of serviceable districts
subject to Max_Vehicles:
sum{i in 1..district} y[i] <=2;
#Ensuring less than 2 vehicles
```

AMPL Data Code:

```
param A: 1 2 3 4 5 6 :=
1 0 4 3 6 6 5
2 4 0 7 5 5 6
3 3 7 0 4 3 5
4 6 5 4 0 7 5
5 6 5 3 7 0 2
6 5 6 5 5 2 0;
```

AMPL Run Code:

```
reset;
model problem4.mod;
data problem4.dat;
option solver gurobi;
solve;
display T;
display y;
display x;
```

AMPL Solution

```
ampl: include run.run;
Gurobi 5.5.0: optimal solution; objective 14
29 simplex iterations
T = 7

y [*] :=
1  0
2  0
3  1
4  0
5  1
6  0
;

x [*,*]
:   1   2   3   4   5   6    :=
1   0   0   1   0   0   0
2   0   0   0   0   1   0
3   0   0   1   0   0   0
4   0   0   1   0   0   0
5   0   0   0   0   1   0
6   0   0   0   0   1   0
;
```

5

The coordinator wants to locate the facilities to minimize the average travel time to the districts, weighted by district population. Formulate this problem as an integer linear program.

A =

*	1	2	3	4	5	6
1	0	4	3	6	6	5
2	4	0	7	5	5	6
3	3	7	0	4	3	5
4	6	5	4	0	7	5
5	6	5	3	7	0	2
6	5	6	5	5	2	0

P = 21 35 15 60 20 37

Let $D = \{1, \cdots, 6\}$ represent the set of all possible districts.

Let $y_j = \begin{cases} 1 & \text{if district } j \text{ is chosen to receive a vehicle} \\ 0 & \text{otherwise} \end{cases}$, for $j \in D$.

Let $x_{ij} = \begin{cases} 1 & \text{if district } i \text{ is served by a vehicle in district } j \\ 0 & \text{otherwise} \end{cases}$, for $j \in D$.

Let a_{ij} represent a position in the incidence matrix A and reflect the time to serve district j from district i.

Let p_i represent the population of district i.

Minimize $\sum_{i \in D} \sum_{j \in D} p_i a_{ij} x_{ij} * \frac{1}{\sum\limits_{i \in D} p_i}$

subject to:

$\sum_{j \in D} x_{ij} = 1$ for $i \in D$

$x_{ij} \leq y_j$ for $i \in D, j \in D$

$y_j \in \{0, 1\}$, for $j \in D$

$x_{ij} \in \{0, 1\}$, for $i \in D$

Based on this formulation, we would make the following recommendation to the coordinator:

Clarke County EMS Coordinator,

If Clarke County has the budget for 2 EMS vehicles and wishes to minimize the average service time of each region, prioritizing by population size, it *is recommended that vehicles be assigned to districts 4 and 6. The vehicle assigned to district 4 should be responsible for servicing districts 2, 3, and 4. The vehicle assigned to district 6 should be responsible for servicing districts 1, 5, and 6. This will minimize the average wait time to approximately 2.02 minutes.*

AMPL Code

AMPL Model Code

```
param district = 6;
param A {i in 1..district, j in 1..district};
param P {i in 1..district};

var y{1..district} binary;
#1 if district j is chosen to receive a vehicle, 0 otherwise
var x{1..district,1..district} binary;
#1 if district i is serviced by district j, 0 otherwise

minimize Average_Time :
(sum{i in 1..district}sum{j in 1..district}
P[i]*A[i,j]*x[i,j])/sum{i in 1..district}P[i];

subject to One_Vehicle {i in 1..district}:
sum{j in 1..district} x[i,j] = 1;
#Making sure that only one vehicle services a district
subject to Max_Districts {i in 1..district, j in 1..district}:
x[i,j] <= y[j];
#Setting Max number of serviceable districts
subject to Max_Vehicles:
sum{i in 1..district} y[i] <=2;
#Ensuring less than 2 vehicles
```

AMPL Data Code

```
param A: 1 2 3 4 5 6 :=
1 0 4 3 6 6 5
2 4 0 7 5 5 6
3 3 7 0 4 3 5
4 6 5 4 0 7 5
5 6 5 3 7 0 2
6 5 6 5 5 2 0;
```

```
param P:= 1 21 2 35 3 15 4 60 5 20 6 37;
```

AMPL Run Code

```
reset;
model problem5.mod;
data problem5.dat;
option solver gurobi;
solve;
display Average_Time;
display y;
display x;
```

AMPL Solution

```
ampl: include run.run;
Gurobi 5.5.0: optimal solution; objective 2.021276596
24 simplex iterations
Average_Time = 2.02128

y [*] :=
1  0
2  0
3  0
4  1
5  0
6  1
;

x [*,*]
:   1   2   3   4   5   6    :=
1   0   0   0   0   0   1
2   0   0   0   1   0   0
3   0   0   0   1   0   0
4   0   0   0   1   0   0
5   0   0   0   0   0   1
6   0   0   0   0   0   1
;
```