Modern Algebra
Homework 10a
Chapter 10
Read 10.1-10.4
Complete 10.8, 10.15, 10.22, 10.29, 10.30

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10.8

This exercise shows that the complex numbers $\mathbb{C}$, containing all the numbers $a+bi$, for $a, b \in \mathbb{R}$, form a field, and explores arithmetic in that field.

a.) *What is the sum $(a+bi)+(c+di)$ of two complex numbers, written in $a+bi$ form?*
$(a+bi)+(c+di) = (a+c)+(b+d)i$.

b.) *What is the product $(a+bi)(c+di)$ of two complex numbers, written in $a+bi$ form?*
$(a+bi)(c+di) = ac+adi+bci+bd = (ac+bd)+(ad+bc)i$.

c.) *The additive inverse of any $a+bi$ is clearly $-a-bi$, which is obviously in the set $\mathbb{C}$. The multiplicative inverse of $a+bi$, however, is $\frac{1}{a+bi}$, and it is not obvious that it is in $\mathbb{C}$. Prove that it is.*
Let $z = a+bi, z \in \mathbb{C}$ with $z^{-1} = \frac{1}{a+bi}$ where $a, b, u, v \in \mathbb{R}$.

We know that $zz^{-1} = 1$, by inverse.

$zz^{-1} = (a+bi)\frac{1}{a+bi} = 1$.

Therefore, the multiplicitave inverse is in $\mathbb{C}$.

d.) *Write all the following powers of i in $a+bi$ form.*

$i = (a+bi)^2$

$i^2 = -1 = \sqrt{(a+bi)^2}$

$i^3 = -i = -(a+bi)^2$

$i^4 = a + bi$

f.) *The number i^n is equal to:*

1, when $n \cong_4 0$

i, when $n \cong_4 1$

-1, when $n \cong_4 2$

$-i$, when $n \cong_4 3$

f.) *Show that $\sqrt{i}$ is the element $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ from $\mathbb{C}$, by showing that the result of squaring that element is i.*

$(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i)0^2 = i$ which implies that $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i = i$.

10.15

Recall the function $\phi : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2})$ from the section 10.5.3 defined by $\phi(a + b\sqrt{2}) = a - b\sqrt{2}$

a.) *Show that the field is an automorphism.*
$\phi(a + b) = a - b = \phi(a)k + \phi(-b) = \phi(a) + \phi(b)$

$\phi(ab) = ab - 0 = ab = (a - 0)(b - 0) = \phi(a)\phi(b)$

b.) *Show why it fixes $\mathbb{Q}$.*
For all $a \in \mathbb{R}$, $\phi(a) = a - 0 = a$. Therefore $\mathbb{Q}$ is fixed by all ϕ.

10.22

Find the irreducible polynomial for each of the following algebraic numbers.

a.) $\sqrt{15} = x^2 - 15$

b.) $\sqrt[5]{1 + \sqrt{2}}$
$x^5 = 1 + \sqrt{2}$
$x^5 - 1 = \sqrt{2}$
$(x^5 - 1)^2 = 2$

$(x^5-1)^2-2=0$

c.) $\sqrt{10}+\sqrt{11}$
$\sqrt{10}=x^2-10,\ \sqrt{11}=x^2-11$

Therefore, $\sqrt{10}+\sqrt{11}=(x^2-10)+(x^2-11)=2x^2-21$

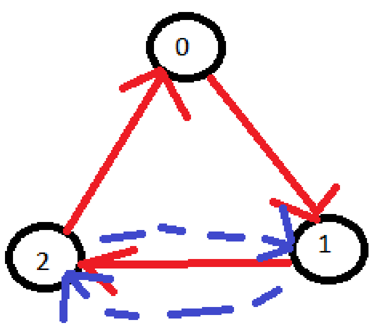
d.) $\sqrt[3]{2}+10$
$\sqrt[3]{2}=x$
$2=x^3$
$x^3-2=0$

Therefore $\sqrt[3]{2}+10=(x^3-2)+10=x^3+8$.

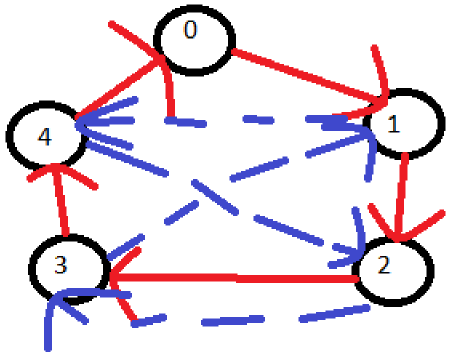
10.29

a.) *Are the addition and multiplication operations in finite fields addition and multiplication mod some number?*
Yes, since by combining the Cayley diagrams, every possible action is covered except the identity for multiplication.

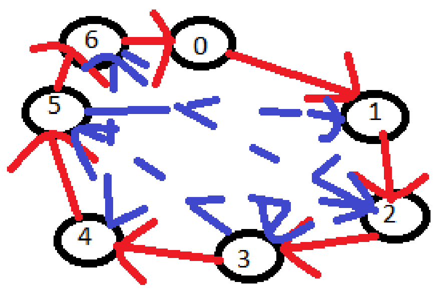
b.) *There is never more than one finite field of a given order. Create the Cayley diagrams for orders 3,5,7, and 11.*



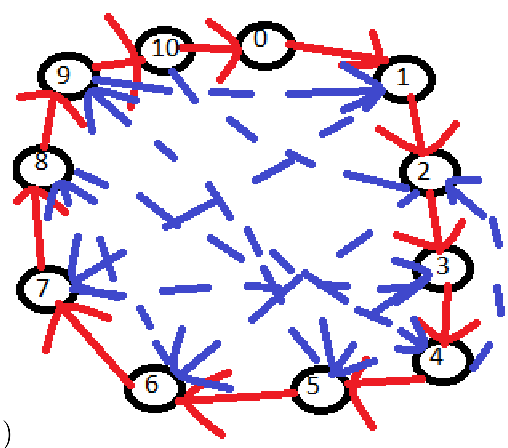
a.)



b.)



c.)



d.)

10.30

Why can the group C_4 under addition not be made into a finite field by overlaying a multiplicative structure on $\{1, 2, 3\}$? Why do C_6 and $C_1 5$ have the same problem?

We cannot make C_4 under addition into a finite field by overlaying a multiplicative structure since the order of a finite field must be a prime number and if we overlay the multiplicative structure, we do not generate a diagram of prime power. The same problem arise for C_6 and $C_1 5$ because of their underlying structure.