CSCI 628
Homework 2

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Problem 1

For the following system of equations

$$x_1 + x_2 - x_3 + x_4 = 1$$
$$x_2 - 2x_3 - x_4 + x_5 = -1$$
$$x_1 + x_3 + x_5 = 2$$

which of the following is a basic solution? Justify your answers.

We know that a **Basic Solution** corresponding to $Ax = b$ corresponding to basis B is the solution to $Ax = b, x_j = 0$ for $j \in N$.

We also know that for ordered subset B of $\{1, \cdots, n\}$ and $A \in \mathbb{R}^{m \times n}$, then B is a **basis** for A if A_B is invertible.

For this system of equations,

$$A = \begin{pmatrix} 1 & 1 & \text{-1} & 1 & 0 \\ 0 & 1 & \text{-2} & \text{-1} & 1 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix} \text{ and b} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

a.) $[2, -1, 0, 0, 0]^T$;

For the given vector, column indices $1, \&2$ must be in the basis, otherwise $x_j \neq 0$ for some $j \in B$.

$$A_B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The matrix A_B is not invertible because it is not square. However, since the number of basic variables is less than $m = 3$, we can use the extentsion tool to construct a basis. We could take $B = \big(1, 2, 5\big)$. Then $A_B =$

$$A_B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

A_B is square and row reduces to the identity matrix. Therefore A_B is an invertible matrix and $\hat{x}^T = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \end{pmatrix}$ is a basic solution.

b.) $[1, 1, 1, 0, 0]^T$;

For the given vector, column indices $1, 2, \&3$ must be in the basis, otherwise $x_j \neq 0$ for some $j \in B$.

$$A_B = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{pmatrix}$$

Since A_B is square, we may check to see if it is invertible.

$$A_B = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{pmatrix} \Bigg| \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We can use elementary row operations to reduce the augmented matrix to:

$$A_B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & 0 \end{pmatrix} \Bigg| \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix}$$

Therefore the matrix is singular(not invertible) and the vector $\hat{x}$ is not a basic solution.

c.)$\frac{1}{3}[11, 0, 0, -2, -5]^T$;

For the given vector, column indices $1, 4, \&5$ must be in the basis, otherwise $x_j \neq 0$ for some $j \in B$.

$$A_B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Since A_B is square, we may check to see if it is invertible.

$$A_B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \Bigg| \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We can use elementary row operations to reduce the augmented matrix to:

$$A_B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \Bigg| \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

Therefore the matrix is singular (not invertible) and $\hat{x}$ is not a basic solution.

d.)$\frac{1}{6}[17, -3, 0, -2, -5]^T$;

For the given vector, column indices $1, 2, 4, \&5$ must be in the basis, otherwise $x_j \neq 0$ for some $j \in B$.

$$A_B = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

The resultant matrix has more columns than there are rows of A and is not square (not invertible). Therefore, $\hat{x}$ is not a basic solution.

Problem 2

Let $A \in \mathbb{R}^{m \times n}$. We said that the columns in A are linearly independent if the only solution to $Ax = 0$ is $x = 0$. Prove the following proposition, which gives an alternative way to define linear independence.

The columns in A are linearly independent if and only if $Ax = b$ has at most one solution for every $b \in \mathbb{R}^m$.

Proof: $\Rightarrow$ If the columns in A are linearly independent, then, by definition, rank A $= n$. If $n \leq m$, then we can use the extention tool and form a basis for $\mathbb{R}^n$ by adding $m-n$ additional independent vectors to the existing column vectors of A. These new vectors will be slack variables. The resultant $m \times m$ matrix, S, is square with rank m and is therefore invertible. This inverted matrix contains the inversion of the original column vectors. We will extract these first n column vectors and denote the extracted matrix A'. Since these were extracted from the inverted S matrix, which was an extension of A, we have $A'A = I_n$.

Now, we must prove that $Ax = b$ has at most one solution. If $Ax = b$, has no solution, it is trivial. So, suppose that $Ax = b$ has two solutions (i.e. more than one). We will call these solutions x_1 and x_2. This implies $x_1 = A'Ax_1 = A'b = A'Ax_2 = x_2$. Therefore, if the columns of A are linearly independent, the system $Ax = b$ can have at most one solution.

$\Leftarrow$If $Ax = b$ has at most one solution for every $b \in \mathbb{R}$, then for $b = 0$, $Ax = b$ has at most one solution. However, $Ax = 0$ always has the trivial solution, $x = 0$. Therefore, $Ax = b$ has only the trivial solution $x = 0$ and the columns of A are linearly independent by definition.

Problem 3

Consider the set of vectors x satisfying

$$\begin{aligned} &x_1 + 3x_2 + 2x_3 + x_4 = 4 \\ &2x_1 + 6x_2 + 5x_3 + x_4 = 9 \\ &x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

Answer the following questions, justifying your answers carefully.

Based on the given system of equations, we can derive the following matrices:

$$\text{A} = \begin{pmatrix} 1 & 3 & 2 & 1 \\ 2 & 6 & 5 & 1 \end{pmatrix} \text{ and b} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}.$$

Therefore, we have $m = 2$ and $n = 4$ and can obtain a basic solution by settin $n - m = 4 - 2$ variables to 0.

a.) Find an extreme point of the feasible region.

We know that the feasible region is a covex set $P := \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$. An extreme point of the feasible region is a vertex of P. Since P is a convex set, this means that the extreme point (vertex) cannot be represented as a convex combination of any other two distinct points of P. Therefore, we know that the set, E, of extreme points of the feasible region P is exactly the set, B, of all basic feasible solutions of the system of equations.

Using elementary row operations, we can bring the augmented matrix A to reduced row echelon form. The resultant matrix is as follows

$$\begin{pmatrix} 1 & 3 & 0 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix}$$

Therefore, we will select x_1 and x_3 as basic variables. Which means that x_2 and x_4 are non-basic.

To determine a basic feasible solution, we must set the non-basic variable coefficents to zero and choose the basic variables such that the contraints are satisfied. We will choose $x_1 = 2, x_3 = 1$.

Thus, an extreme point of the feasible region is $\hat{x} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$.

We can verify that $\hat{x}$ is feasible by plugging it into the original system of equations.

$$\begin{aligned} &1 * 2 + 3 * 0 + 2 * 1 + 0 = 4 \\ &2 * 1 + 6 * 0 + 5 * 1 + 0 = 9 \\ &x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

b.) Find a basic solution that is not feasible.

We have determined in part A that the basic variables are x_1 and x_3. To find a basic solution that is not feasible, we must find a solution that violates the constraints.

A basic solution which isn't feasible is $\hat{x} = \begin{pmatrix} 0 \\ 2 \\ -1 \\ 0 \end{pmatrix}$.

It is basic because the number of non-zero components is less than or equal to $m = 2$.

We can verify that $\hat{x}$ is a basic by plugging it into the original system of equations.

$$\begin{aligned} 1 * 0 + 3 * 2 + 2 * -1 + 0 = 4 \\ 2 * 0 + 6 * 2 + 5 * -1 + 0 = 9 \end{aligned}$$

However, the basic solution is not feasible because $x_3 \leq 0$.

c.) Find a feasible solution that is not basic.

The following solution isn't basic since it has more than $m = 2$ non-zero components.

$$\hat{x} = \begin{pmatrix} 1 \\ \frac{1}{3} \\ 1 \\ 0 \end{pmatrix}$$

We can verify that $\hat{x}$ is feasible by plugging it into the original system of equations.

$$\begin{aligned} & 1 * 1 + 3 * \tfrac{1}{3} + 2 * 1 + 0 = 4 \\ & 2 * 1 + 6 * \tfrac{1}{3} + 5 * 1 + 0 = 9 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

d.)Find a feasible solution that is not an extreme point: justify your answer by using the definition of extreme point (so you're not allowed to use any other results that we have proved in lecture).

Extreme Point:$\hat{x}$ is an extreme point of F if there exists no $y, z \in F$ such that $y \neq \hat{x}$ and $z \neq \hat{x}$ and $\lambda \in (0,1)$ such that $\hat{x} = \lambda y + (1-\lambda)z$.

Let $y = \hat{x} + \epsilon d$ and $x = \hat{x} + \epsilon d$. Then $\hat{x} = \frac{1}{2}y + \frac{1}{2}z$ and $z_i = 0 = y_i$ if $\hat{x} = 0$ since $d_j = 0$. If ϵ is small enough, then $z_i, y_i \geq 0$ if $x_i \geq 0$.

$$\text{A} = \begin{pmatrix} 1 & 3 & 2 & 1 \\ 2 & 6 & 5 & 1 \end{pmatrix} \text{ and b} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}. \text{ and let } \hat{B} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

We have shown that $\hat{x} = \begin{pmatrix} 1 \\ \frac{1}{3} \\ 1 \\ 0 \end{pmatrix}$ is a feasible solution that is not basic.

Which implies that $A_{\hat{B}} = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 6 & 5 \end{pmatrix}$.

We must solve the following equation for $\hat{d}$: $A_{\hat{B}}\hat{d} = 0$.

This implies that $\hat{d} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$

$$y = \hat{x} + \epsilon\hat{d} = \begin{pmatrix} 1 \\ \frac{1}{3} \\ 1 \end{pmatrix} + \epsilon \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+3\epsilon \\ \frac{1}{3} - \epsilon \\ 1 \end{pmatrix}$$

$$z = \hat{x} - \epsilon\hat{d} = \begin{pmatrix} 1 \\ \frac{1}{3} \\ 1 \end{pmatrix} - \epsilon \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1-3\epsilon \\ \frac{1}{3} + \epsilon \\ 1 \end{pmatrix}$$

We will choose ϵ such that $\epsilon \in (0,1)$ and therefore satifies the conditions on λ in the extreme point definition. We will let $\epsilon = \frac{1}{6}$. Therefore, $y = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{6} \\ 1 \end{pmatrix}$,

$$z = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}.$$

Based on our equations for y and z, we have $\hat{x} = \frac{1}{2}y + \frac{1}{2}z = \frac{1}{2}\begin{pmatrix} \frac{3}{2} \\ \frac{1}{6} \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix} =$

$$\begin{pmatrix} \frac{3}{4} \\ \frac{1}{12} \\ \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{3} \\ 1 \end{pmatrix}$$

According to the definition of extreme point, this $\hat{x}$ is not an extreme point. It is feasible because it solves the system of equations and maintains the positivity constraint. Again, it is non-basic because it has more than $m = 2$ non-zero components and its feasibility can be demonstrated by checking the original system of equations.

Problem 4

A European country in which frogs are considered a delicous food buys oil from 3 Middle-East countries (call them A, B, C) and stores it in 4 main locations (call them 1, 2, 3, 4). Each one of the Middle-East countries limits the number of barrels it is willing to sell to the European country to 250K, 500K, and 600K per month (respectively). In order to satisfy the country's needs, quatities of at least 100k, 200k, 300k, and 400k barrels should be provided every month to locations 1, 2, 3, 4 respectively. The per barrel prices (in Euro's) for buying and shipping from each country to each location are given in the following table:

	Country A	Country B	Country C
Location 1	12	9	10
Location 2	10	11	14
Location 3	8	11	13
Location 4	11	13	9

a.) Assume that fractional barrels can be provided. Write a linear program that finds how many barrels should be bought from each country and to what locations they should be sent so that the total cost is minimized.

$i = 1, 2, 3$: Countries
$j = 1, 2, 3, 4$: Locations
$x_{i,j}$: Barrels of oil sent from country i to location j
$C_{i,j}$: Cost of shipping barrels from country i to location j
S_i: Supply at country i
D_j: Demand at country j

minimize cost: $\sum_{i=1}^{3}\sum_{j=1}^{4} c_{i,j}x_{i,j}$

s.t. $\sum_{j=1}^{4} x_{i,j} = S_i$, $i = 1, 2, 3$

$\sum_{i=1}^{3} x_{i,j} = D_j$, $j = 1, 2, 3, 4$

$x_{i,j} \geq 0$, for $i = 1, 2, 3$, $j = 1, 2, 3, 4$

b.)Now think about a case in which the European country does not want any location to be highly dependent on supply from a specific country. So it is required in addition that the supply from a certain country to a certain location will be at least $\frac{1}{4}$ and no more than $\frac{1}{2}$ the total supply to that location. Adapt your LP to capture the new requirement. Can the total cost decrease/increase? Explain.

$i = 1, 2, 3$: Countries
$j = 1, 2, 3, 4$: Locations
$x_{i,j}$: Barrels of oil sent from country i to location j
$C_{i,j}$: Cost of shipping barrels from country i to location j
S_i: Supply at country i
D_j: Demand at country j

minimize cost: $\sum_{i=1}^{3}\sum_{j=1}^{4} c_{i,j}x_{i,j}$

s.t. $\sum_{j=1}^{4} x_{i,j} = S_i$, $i = 1, 2, 3$

$\sum_{i=1}^{3} x_{i,j} = D_j,\ j = 1, 2, 3, 4$
$x_{i,j} \geq 0$, for $i = 1, 2, 3,\ j = 1, 2, 3, 4$

$-x_{i,j} \leq -\frac{1}{4}D_j$ for $i = 1, 2, 3$ and for $j = 1, 2, 3, 4$.
$x_{i,j} \leq \frac{1}{2}D_j$ for $i = 1, 2, 3$ and for $j = 1, 2, 3, 4$.

The total cost cannot decrease since we are adding constraints and the previous LP was already a maximization problem. The total cost can, however, increase in order to accomodate these new restrictions. In our previous problem, we had no base requirments on the amount to purchase from a country; we could elect to purchase nothing if it was optimal. Since we are now required to purchase at least $\frac{1}{4}$ of the total demand at a location from a country, we may make a previously non-optimal purchase and increase cost.

c.)In this question, you will build an AMPL model and data file to solve part (a). You will not have to build it from scratch since the problem is similar to the transportation problem, for which a sample model and data file are included in the MODELS directory that was part of the zipfile downloaded when AMPL was installed. Copy the model file transp.mod and the data file trasnp.dat from the MODELS folder to your working folder. If you need help understanding the model, read through section 3.1-3.2 of the AMPL book (available at http://www.ampl.com/OOK/CHAPTERS/06-tut3.pdf). You will see that the model file contains a line which checks that the input data satisfies that the total supply is equal to the tota demand which is not the case in our oil example, so remove the line "check: sum{ i in ORIG } supply[i] = sum {j in DEST }demand[j];". Make any additional changes necessary so that you obtain the model formulated in part (a). Next open the data file transp.dat and modify it so it contains the data for our problem. Solve the problem using AMPL (any solver is OK). Report to the optimal value, and the amount to be shipped from each country to each location, and email your .mod file and .dat file.

Based on the formulation, the minimum total cost is 9250k. To achieve this cost, country A supplies location 3 with 250k. Country B supplies locations 1, 2, and 3 with 100k, 200k, and 50k, respectively. Country C supplies location

4 with 400k. AMPL Model:

```
set ORIG;   # origin
set DEST;   # destinations

param supply {ORIG} >= 0;
# amounts available at origins
param demand {DEST} >= 0;
# amounts required at destinations

param cost {ORIG,DEST} >= 0;
# shipment costs per unit
var Trans {ORIG,DEST} >= 0;
# units to be shipped

minimize Total_Cost:
   sum {i in ORIG, j in DEST} cost[i,j] * Trans[i,j];

subject to Supply {i in ORIG}:
   sum {j in DEST} Trans[i,j] <= supply[i];

subject to Demand {j in DEST}:
   sum {i in ORIG} Trans[i,j] >= demand[j];
```

AMPL Data:

```
data;

param: ORIG:  supply :=
# defines set "ORIG" and param "supply"
        A   250
        B   500
        C   600;

param: DEST:  demand :=
# defines "DEST" and "demand"
        One      100
```

```
        Two      200
        Three    300
        Four     400;

param cost:
One Two Three Four :=
A 12 10 8 11
B 9 11 11 13
C 10 14 13 9;
```

AMPL Output:

```
ampl: model transp.mod;
ampl: option solver gurobi;
ampl: data transp.dat;
ampl: solve;
Gurobi 5.6.3: optimal solution; objective 9250
6 simplex iterations
ampl: display Trans;
Trans :=
A Four      0
A One       0
A Three   250
A Two       0
B Four      0
B One     100
B Three    50
B Two     200
C Four    400
C One       0
C Three     0
C Two       0
;
```