CSCI 628
Homework 3

Megan Rose Bryant
Department of Mathematics
William and Mary

September 25, 2014

1: Interpreting Tableaus

The following is a tableau for a linear program in standard equality form.

$$\begin{array}{llllll} z & +\frac{1}{3}x_1 & & +\frac{1}{3}x_3 & & = 8 \\ & \frac{1}{3}x_1 & & -\frac{2}{3}x_3 & +x_4 & = 1 \\ & \frac{1}{3}x_1 & +x_2 & -\frac{1}{6}x_3 & & = 3 \end{array}$$

a.) Write down the corresponding basis.

The corresponding basis is $B = [2, 4]$.

b.) Write down an optimal solution for the original problem, and its objective value.

An optimal solution is

$$\hat{x} = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix}.$$

The corresponding objective value is 8.

c.) How would you explain to a freshman (who understands equations but not linear programming) why the solution you wrote down in art (b) is feasible for the original linear program (whatever it is)?

I would say that the solution is feasible because when you set the nonbasic variables to zero, the corresponding solution is positive. An infeasible solution would have at least one negative value.

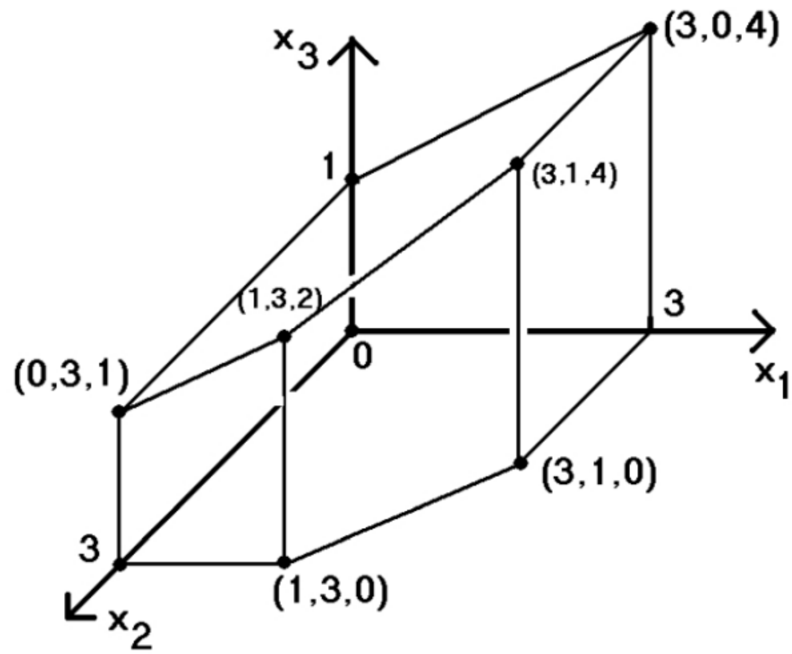
d.) In the same style as part (c), explain why your solution in part (b) is optimal.

I would explain that this solution is optimal because we cannot increase the objective function any further. We know this because both of the coefficients in the $z = \frac{-1}{3}x_1 - \frac{1}{3}$ equation are negative and increasing them would subtract from the optimal value.

2: Geometry of linear programming in 3-D

The next page has a picture of the feasible region for the following linear program.

$$
\begin{array}{lrrrl}
\max & x_1 & +2x_2 & +x_3 & \\
\text{s.t.} & x_1 & & & \leq 3 \\
 & & x_2 & & \leq 3 \\
 & x_1 & +x_2 & & \leq 4 \\
 & -x_1 & & +x_3 & \leq 1 \\
 & x_1, & x_2, & x_3 & \geq 0
\end{array}
$$



Solve the linear program using the simplex method. First, introduce slack variables and use them as the basic variables in an initial feasible tableau. Among the possible entering variables, choose one with largest reduced cost (most negative objective coefficient) at each iteration. Plot the path taken by the simplex method on the picture. Does your final tableau imply that the optimal solution is unique. If not, can you find another optimal solution?

$$
\begin{array}{lllllll}
z = & -x_1 - 2x_2 & -x_3 & & & & = 0 \\
& x_1 & & + S_1 & & & = 3 \\
& + x_2 & & & +S_2 & & = 3 \\
& x_1 + x_2 & & & & + S_3 & = 4 \\
& -x_1 & +x_3 & & & & +S_4 = 1
\end{array}
$$

Our starting basis is $B = [4, 5, 6, 7]$. We will select the variable with the largest reduced cost (most negative objective coefficient) to enter the basis. The variable we have selected is x_2. This results in the following updated tableau.

$$
\begin{array}{llllll}
z = & -x_1 & -x_3 & & +2S_2 & = 6 \\
& x_1 & & + S_1 & & = 3 \\
& + x_2 & & & +S_2 & = 3 \\
& x_1 & & & -S_2 + S_3 & = 1 \\
& -x1 & +x_3 & & & +S_4 = 1
\end{array}
$$

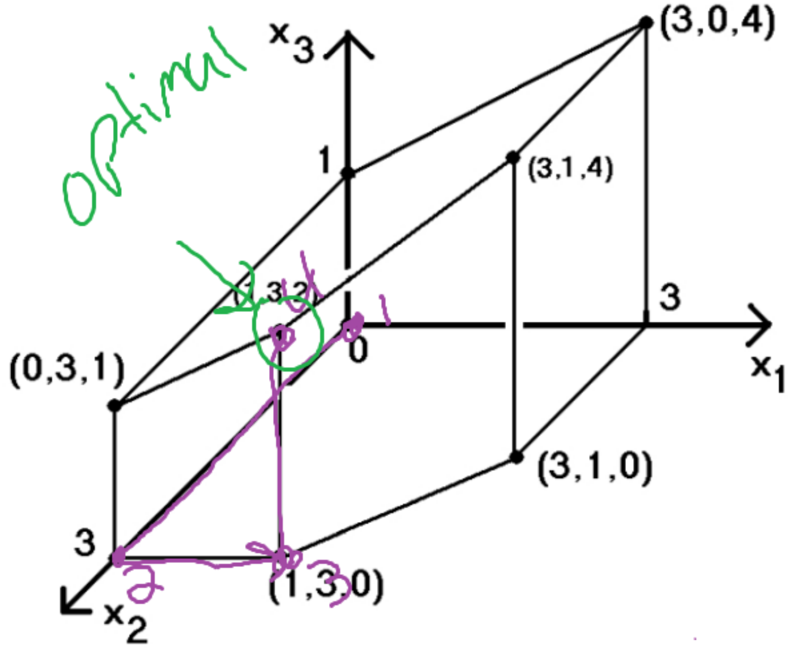
Our new basis is $B[2, 4, 6, 7]$. Since the variables x_1 and x_2 both have a reduced cost of 1, we can select either to enter the basis. We will select x_1 to enter the basis. This results in the following updated tableau.

$$
\begin{array}{lllllll}
z = & & & -x_3 & & +S_2 + S_3 & = 7 \\
& & & & + S_1 & +S_2 - S_3 & = 2 \\
& & + x_2 & & & +S_2 & = 3 \\
& x_1 & & & & -S_2 + S_3 & = 1 \\
& & & +x_3 & & -S_2 + S_3 & +S_4 = 2
\end{array}
$$

Our new basis is $B = [1, 2, 4, 7]$. The only variable with a negative coefficiant remaining is x_3. Therefore, we will select x_3 to enter the basis. This results in the following updated tableau.

$$
\begin{array}{lllllllll}
z = & & & & & & +2S_3 & +S_4 & = 9 \\
& & & & +S_1 & +S_2 & -S_3 & & = 2 \\
& & +x_2 & & & +S_2 & & & = 3 \\
& x_1 & & & & -S_2 & +S_3 & & = 1 \\
& & & +x_3 & & -S_2 & +S_3 & +S_4 & = 2
\end{array}
$$

We have arrived at the opimal tableau since all of the coefficients in the Z-equation are positive. We have an optimal ojective value of 9 and an optimal solution of $\hat{x}^T = \begin{pmatrix} 1 & 3 & 2 & 2 & 0 & 0 & 0 \end{pmatrix}$



The final tableau implies that the optimal solution is not unique (i.e. there exists alternate optimal solutions). This is because the variable S_2 has a reduced cost of zero. We can find an alternate optimal solution as follows:

$$
\begin{gathered}
x_1 = 1 + S_2 - S_3 \\
x_2 = 3 - S_2 \\
x_3 = 2 + S_2 - S_3 - S_4
\end{gathered}
$$

$$S_1 = 2 - S_2 + S_3$$

S_3 and S_4 are stil going to be nonbasic, so they will be set to zero. We will allso parameterize the equations with $S_3 = t$.

$$\begin{aligned} x_1 &= 1 + t \\ x_2 &= 3 - t \\ x_3 &= 2 + t \\ S_1 &= 2 - t \end{aligned}$$

Therefore, we know that the set of all optimal nonbasic solutions is the points $\hat{x}^T = \begin{pmatrix}(1+t) & (3-t) & (2+t) & (2-t) & t & 0 & 0\end{pmatrix}, 0 \leq t \leq 2$.

For a solution to be basic as well as feasible and optimal, a variable must leave the basis when s_2 enters. Therefore, let $t = 2$ and an alternative optimal solution would be $\hat{x}^T = \begin{pmatrix}3 & 1 & 4 & 0 & 2 & 0 & 0\end{pmatrix}$, which yields an objective function of $z = 9$.

b.)Now try solving the new problem obtained by replacing the objective function $x_1 + 2x_2 + x_3$ by $x_1 + x_2 + x_3$. This time, at each iteration, among the possible entering variables, choose the one with the smallest entering subscript. Again, plot the path, and again decide if the optimal solution is unique.

$$\begin{array}{llllllll} z = & -x_1 - x_2 & -x_3 & & & & & = 0 \\ & x_1 & & + S_1 & & & & = 3 \\ & \quad + x_2 & & & +S_2 & & & = 3 \\ & x_1 + x_2 & & & & + S_3 & & = 4 \\ & -x_1 & +x_3 & & & & +S_4 & = 1 \end{array}$$

Our starting basis is once again $B = [4, 5, 6, 7]$. We will choose the smallest subscript to enter the basis and thus select x_1. The following is the resulting tableau.

$$
\begin{array}{llllll}
z = & & -x_2 & -x_3 + S_1 & & = 3 \\
& x_1 & & + S_1 & & = 3 \\
& & + x_2 & & +S_2 & = 3 \\
& & + x_2 & - S_1 & + S_3 & = 1 \\
& & & +x_3 + S_1 & & +S_4 = 4
\end{array}
$$

Our basis is now $B = [1, 5, 6, 7]$. We will choose the smallest subscript remaining to enter the basis and thus select x_2. The following is the resulting tableau.

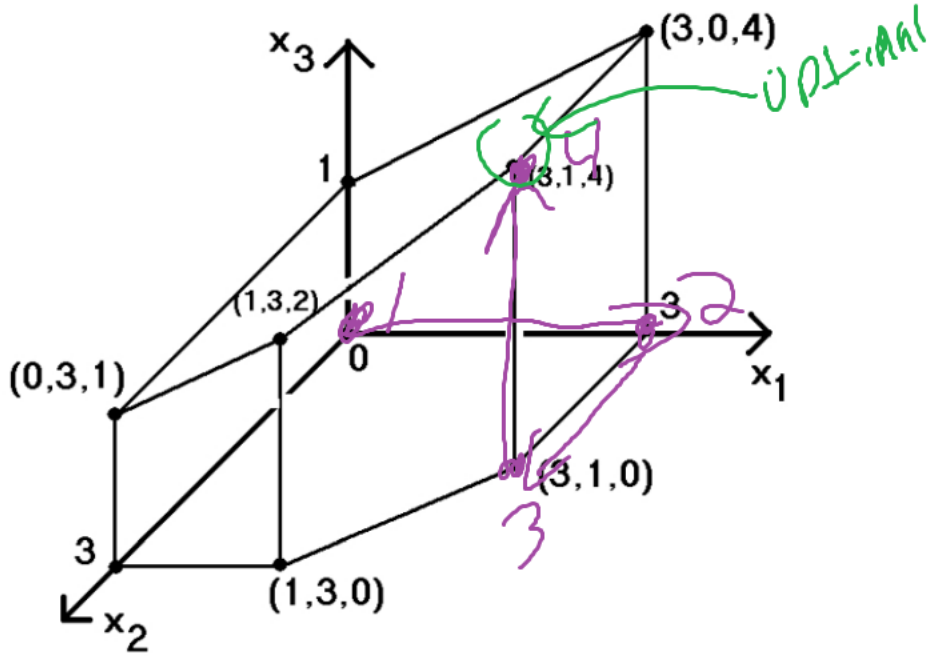
$$
\begin{array}{llllll}
z = & & & -x_3 & + S_3 & = 4 \\
& x_1 & & + S_1 & & = 3 \\
& & & + S_1 & +S_2 - S_3 & = 2 \\
& & + x_2 & - S_1 & + S_3 & = 1 \\
& & & +x_3 + S_1 & & +S_4 = 4
\end{array}
$$

Our basis is now $B = [1, 2, 5, 7]$. We will choose the smallest subscript remaining to enter the basis and thus select x_3. The following is the resulting tableau.

$$
\begin{array}{llllll}
z = & & & + S_1 & + S_3 & +S_4 = 8 \\
& x_1 & & + S_1 & & = 3 \\
& & & + S_1 & +S_2 - S_3 & = 2 \\
& & + x_2 & - S_1 & + S_3 & = 1 \\
& & & +x_3 + S_1 & & +S_4 = 4
\end{array}
$$

The resulting tableau is optimal since there are no more negative coefficients in the z equation. The optimal basis is $B = [1, 2, 3, 5]$. The optimal basic solution is $\hat{x}^T = \begin{pmatrix} 3 & 1 & 4 & 0 & 2 & 0 & 0 \end{pmatrix}$. The optimal objective value is 8.

Since there all of the non basic variables have reduced costs other than 0, there are no alternative solutions and the optimal solution is unique.



3: Entering & leaving variable pairs

Consider the following tableau for a linear program in standard equality form

z			$+4x_3$		$-3x_5$	$-2x_6$	$= 5$
			$4x_3$	x_4	$+3x_5$	$+2x_6$	$= 6$
		x_2	$+2x_3$		$-x_5$	$+2x_6$	$= 4$
	x_1		$-2x_3$		$+2x_5$	$+x_6$	$= 2$

List all pairs (k, r) such that k could be the entering index and r could be the leaving index for an interation of the simplex method from this tableau.

$(5, 4)$ is a valid pair because all basic variables remain nonnegative.
$(5, 2)$ isn't a valid pair because the objective function would become negative.
$(5, 1)$ is a valid pair because all basic variables remain nonnegative.
$(6, 4)$ isn't a valid pair because x_2 would become negative.
$(6, 2)$ is a valid pair because all basic variables remain nonnegative.
$(6, 1)$ is a valik pair because all basic variables remain nonnegative.

Therefore, the valid pairs that could be the entering and leaving indices are $(5,4),(5,1),(6,2),(6,1)$.

4: Modeling

SailCo must determine how many sailboats to produce in each of next quarters. They must have enough boats available to meet the demand in each quarter:

	Q1	Q2	Q3	Q4
demand	40	60	75	25

Boats made in a quarter can be used to meet demand in the same quarter, or they can be carried over in inventory to meet the demands of later quarters (incurring an inventory cost). The cost of producing a boat is \$400 *for the first* 40 *boats in a quarter and an additional* \$450 *for additional boats (since additional boats will be made in overtime). The cost of keeping a boat in inventory is* \$20/*boat/quarter for boats on hand at the end of a quarter (after production has occurred and demand is satisfied). The initial inventory at the start of Q1 is 10 sailboats. So, for example, if we were to produce 100 boats in Q1, 0 in Q2, 65 in Q3, and 25 in Q4, then the total production cost would be* \$80,250 *and the total inventory cost would be* \$1600. *Formulate an IP to determine a production schedule to minimize the sum of the production and inventory costs. Then, implement your IP in AMPL. Hand in your .mod file and report the optimal objective value plus the optimal production plan.*

x_t = the number of sailboats produced during period t at regular production cost (P), $t = 1,2,3,4$
y_t = the number of sailboats produced during period t at overtime production cost (O), $t = 1,2,3,3$
i_t = the number of sailboats in inventory at the end of quarter t, $t = 1,2,3,4$
P = the cost of producing a sailboat at regular production cost (\$400).
O = the cost of producing a sailboat at overtime production cost (\$450).
H = the cost of holding a sailboat in inventory at the end of a quarter (\$20).
d_t = the demand for sailboats in quarter t, $t = 1,2,3,4$

Minimize $z = P\sum_{t=1}^{4} x_t + O\sum_{t=1}^{4} y_t + H\sum_{t=1}^{4} i_t$
s.t. $i_t = i_{t-1} + x_t + y_t - d_t$, for $t = 1,2,3,4$
$x_t \leq 40$, for $t = 1,2,3,4$
$x_t, y_t.i_t \geq 0$, for $t = 1,2,3,4$

Using AMPL, we arrive at the following optimal objective (minimized total cost): $\$78,450$.

AMPL .mod File:

```
# SailCo Production Optimization

set QUARTERS = {1,2,3,4} ;
set INVQUARTERS = {0,1,2,3,4};

param demand {QUARTERS} >= 0;

param prodlim = 40;

param regcost = 400; # cost for the first prodlim boats of the quarter
param overcost = 450; # cost for additional boats
param invcost = 20; # cost for keeping a boat in inventory until the next quarter

param init_inv = 10; # initial inventory

var RegProd {QUARTERS} >= 0, integer; #regular production units
var OverProd {QUARTERS} >= 0, integer; #overtime production units
var Inv {INVQUARTERS} >= 0, integer; #inventory in stock at end of quarter

# your variables, objective function and constraints go here
minimize Total_Cost:
regcost * ( sum { t in QUARTERS } RegProd[t] )
+ overcost * (sum {t in QUARTERS } OverProd[t] )
+ invcost * (sum {t in QUARTERS} Inv[t]);

subject to Initial_Inventory:
Inv[0] = 10;

subject to End_of_Quarter_Inventory {t in QUARTERS}:
Inv[t] = Inv[t-1] + RegProd[t] + OverProd[t] - demand[t];

subject to Max_Reg_Prod {t in QUARTERS}:
RegProd[t] <= 40;


data;

param demand:=
1        40
2        60
3        75
4        25;
```

AMPL Output:

```
ampl: model sailco.mod;
ampl: option solver gurobi;
ampl: solve;
Gurobi 5.6.3: optimal solution; objective 78450
6 simplex iterations
ampl: display RegProd;
RegProd [*] :=
1  40
2  40
3  40
4  25
;

ampl: display OverProd;
OverProd [*] :=
1   0
2  10
3  35
4   0
;

ampl: display Inv;
Inv [*] :=
0  10
1  10
2   0
3   0
4   0
;
```