LINEAR PROGRAMMING

Homework 5

Fall 2014 | Csci 628 | Megan Rose Bryant

Due date: October 9, in class

0. *Read Sections 9 and 11 in the Course Packet.*

1. *Find an example of a tableau for a linear program in standard equality form, such that at least one of the reduced costs is strictly positive, and yet the corresponding basic solution is optimal.*

   The following tableau is a linear program that is in standard equality form.

$$
\begin{array}{rccrr}
z - 2x_1 & & & +x_4 = & 4 \\
-\,1x_1 & x_2 & & -x_4 = & 2 \\
-\,2x_1 & & +\,x_3 & -x_4 = & 2
\end{array}
$$

   This tableau is optimal with

$$
\begin{aligned}
z &= 4 \\
B &= \begin{pmatrix} 2 & 3 \end{pmatrix} \\
x^T &= \begin{pmatrix} 0 & 2 & 2 & 0 \end{pmatrix}
\end{aligned}
$$

   We see that while the reduced cost of x_2 is $-(-2)$, which is strictly positive, the tableau is optimal because there is not a choice for an entering variable since both of the leading coefficients of the contraints are negative. Also, x_4 is not a candidate to enter the basis because the reduced cost is strictly negative.

   Therefore, we have a tableau with at least one of the reduced costs strictly positive while the corresponding solution is optimal.

2. (a) *Introduce slack variables in the linear program below. Then solve it using the simplex method. Use the slacks as the initial basic variables. At each iteration, use the smallest subscript rule: whenever the simplex method allows you a choice between several possible entering variables, choose the one with the smallest subscript, and use the same rule when the method allows you a choice between possible leaving variables.*

$$
\begin{array}{lrcrcrcl}
\text{maximize} & & & & & 3x_3 & & \\
\text{subject to} & & & x_2 & + & x_3 & \leq & 4 \\
 & & - & x_2 & + & x_3 & \leq & 0 \\
 & x_1 & & & + & x_3 & \leq & 4 \\
 & -x_1 & & & + & x_3 & \leq & 0 \\
 & x_1 & , & x_2 & , & x_3 & \geq & 0.
\end{array}
$$

You need to report for each iteration: (i) the basis B, (ii) the basic feasible solution x and objective value z, (iii) the choices for the entering variable (and which one you choose), and (iv) the min-ratio test for the leaving variable (and which variable you choose to leave the basis).If you use an online pivot tool to do the computations, feel free to leave out the tableaus and only report (i)-(iv).

Starting Tableau:

$$
\begin{array}{lllll}
z & -3x_3 & & & = 0 \\
 & x_2 + x_3 & s_1 & & = 4 \\
 & -x_2 + x_3 & \quad s_2 & & = 0 \\
x_1 & +x_3 & & s_3 & = 4 \\
-x_1 & +x_3 & & \quad s_4 & = 0
\end{array}
$$

$$
\begin{aligned}
z &= 0 \\
B &= \begin{pmatrix} 4 & 5 & 6 & 7 \end{pmatrix} \\
x^T &= \begin{pmatrix} 0 & 0 & 0 & 4 & 0 & 4 & 0 \end{pmatrix} \\
\text{Entering Variable} &: x_3 \\
\text{Min Ratio Test} &: \begin{matrix} s_1 & \frac{4}{1} \\ s_2 & \frac{0}{1} \\ s_3 & \frac{4}{1} \\ s_4 & \frac{0}{1} \end{matrix} \\
\text{Leaving Variable} &: s_2
\end{aligned}
$$

Pivot 1:

$$
\begin{array}{lllll}
z & -3x_2 & +3s_2 & & = 0 \\
 & 2x_2 & s_1 - s_2 & & = 4 \\
 & -x_2 + x_3 & +s_2 & & = 0 \\
x_1 & x_2 & -s_2 & s_3 & = 4 \\
-x_1 & x_2 & -s_2 & \quad s_4 & = 0
\end{array}
$$

$$
\begin{aligned}
z &= 0 \\
B &= \begin{pmatrix} 3 & 4 & 6 & 7 \end{pmatrix} \\
x^T &= \begin{pmatrix} 0 & 0 & 0 & 4 & 0 & 4 & 0 \end{pmatrix} \\
\text{Entering Variable} &: x_2 \\
\text{Min Ratio Test} &: \begin{matrix} s_1 & \frac{4}{2} \\ s_3 & \frac{4}{1} \\ s_4 & \frac{0}{1} \end{matrix} \\
\text{Leaving Variable} &: s_4
\end{aligned}
$$

Pivot 2:

$$\begin{aligned}
&z - 3x_1 & & & & + 3s_4 & &= 0\\
&2x_1 & & & +s_1 + s_2 & - 2s_4 & &= 4\\
&- x_1 & & + x_3 & & + s_4 & &= 0\\
&2x_1 & & & & +s_3 - s_4 & &4\\
&- x_1 & +x_2 & & - s_2 & + s_4 & &= 0
\end{aligned}$$

$$\begin{aligned}
z &= 0\\
B &= \begin{pmatrix}2 & 3 & 4 & 6\end{pmatrix}\\
x^T &= \begin{pmatrix}0 & 0 & 0 & 4 & 0 & 4 & 0\end{pmatrix}\\
\text{Entering Variable} &: x_1\\
\text{Min Ratio Test} &: \begin{matrix} s_1 & \frac{4}{2}\\ s_3 & \frac{4}{2}\end{matrix}\\
\text{Leaving Variable} &: s_1(\text{Lower Subscript})
\end{aligned}$$

Pivot 3:

$$\begin{aligned}
&z & & & +1.5s_1 + 1.5s_2 & & &= 6\\
&1x_1 & & & +0.5s_1 + 0.5s_2 & & - 1s_4 &= 2\\
& & & 1x_3 & +0.5s_1 + 0.5s_2 & & &= 2\\
& & & & -1s_1 - 1s_2 & & 1s_3 + 1s_4 &= 0\\
& & 1x_2 & & +0.5s_1 - 0.5s_2 & & &= 2
\end{aligned}$$

$$\begin{aligned}
z &= 6\\
B &= \begin{pmatrix}1 & 2 & 3 & 6\end{pmatrix}\\
x^T &= \begin{pmatrix}2 & 2 & 2 & 0 & 0 & 0 & 0\end{pmatrix}\\
\text{Entering Variable} &: \text{NA}\\
\text{Min Ratio Test} &: \text{NA}\\
\text{Leaving Variable} &: \text{NA}
\end{aligned}$$

We know that since we began with a maximization problem and the reduced costs in the objective function are all nonpositive that we have reached our optimal tableau. Our optimal objective function value is 6, the associated solution is $x^T = \begin{pmatrix}2 & 2 & 2 & 0 & 0 & 0 & 0\end{pmatrix}$, and the optimal basis is $B = \begin{pmatrix}1 & 2 & 3 & 6\end{pmatrix}$.

(b) *At which iterations was the basic feasible solution degenerate?*

We know that a basic feasible solution is degenerate if one of the basic variables has a value of zero. This occurs in the starting tableau, pivot 1, pivot 2, and pivot 3. In the starting tableau, we have basic variables s_2 and s_4 with associated values of zero. In the tableau resulting from pivot 1, we have basic variables x_3, and s_4 with associated values of zero. In the tableau resulting from pivot 2, we have basic variables x_2 and x_3 with associated values of zero. In the tableau resulting from pivot 3 (the optimal), we have the basic variable s_3 with an associated value of zero. Therefore, every tableau resulting from this LP has a basic feasible solution which is degenerate.

(c) *Which of the pivots you performed are degenerate?*

We know that a pivot is degenerate if the associated solution and objective function value remain the same.. This was the case with both pivot 1 and pivot 2.

For pivot 1, the starting objective function value and associated solutions were $z = 0$ and $x^T = \begin{pmatrix} 0 & 0 & 0 & 4 & 0 & 4 & 0 \end{pmatrix}$. The ending objective function value and associated solutions were $z = 0$ and $x^T = \begin{pmatrix} 0 & 0 & 0 & 4 & 0 & 4 & 0 \end{pmatrix}$. Therefore this pivot was degenerate.

For pivot 2, the starting objective function value and associated solutions were $z = 0$ and $x^T = \begin{pmatrix} 0 & 0 & 0 & 4 & 0 & 4 & 0 \end{pmatrix}$. The ending objective function value and associated solutions were $z = 0$ and $x^T = \begin{pmatrix} 0 & 0 & 0 & 4 & 0 & 4 & 0 \end{pmatrix}$. Therefore this pivot was degenerate.

(d) *Sketch a three-dimensional picture of the feasible region.*

(e) *Which, if any, of the four $\leq$ constraints is redundant? In other words, which, if any, constraint could you delete without changing the feasible region?*

As we can see in our 3-D sketch, none of the constraints are redundant and none of them can be deleted without changing the feasible region. This is because all of the degenerate pivots occur where the constraint lines meet with the positivity requirements; there is no instance in which there are three of more constraints intersecting. Therefore, we would keep all of our existing constraints to maintain the same feasible region.

3. *The tableau for basis B is degenerate if $\bar{b}_i = 0$ for some $i \in B$. A pivot is degenerate if the associated solution and objective value after the pivot is the same as before the pivot.The following claims concern a single pivot of the simplex method, from a current tableau to a new tableau. State whether each claim is true or false. Justify your answers: if you state a claim is true, explain why; if you state it is false, give an example illustrating that it can fail.*

(a) *A degenerate pivot must start from a degenerate current tableau.* **True**. The given definition of a degenerate pivot is a pivot in which the associated solution and objective function value do not change after the pivot is executed. This is equivalent to saying that the basic variable in the associated pivot row (the leaving variable) had an associated value of zero, since that is the only way that the objective function value wouldn't have changed. If the leaving basic variable had an associated value of zero in the starting tableau, then that tableau was degenerate by definition since one of the basic variables had an associated value of zero. Thus, it is true that a degenerate pivot must start from a degenerate current tableau.

(b) *After a degenerate pivot, the new tableau must be degenerate.* **True.** The given definition of a degenerate pivot is a pivot in which the associated solution and objective function value do not change between the starting and ending tableau. We have previously shown in part a.) that a degenerate pivot must start from a degenerate tableau since in order to to maintain the same objective funciton value, the leaving basic variable had an associated value of zero. The second half of the degenerate pivot definition tells us that the solution associated with that stagnant objective function value must also remain the same. The associated solution of the starting tableau must have had at least one basic variable with the associated value of zero by definition of a degenerative tableau. We also know that all nonbasic variables must have had associated values of zero. This implies that if there were m constraints that there are less than m nonzero values in the associated solution. Since the associated solution doesn't change with a degenerative pivot, this condition is maintained in the ending tableau. Therefore, that tableau must be, by definition, degenerate since at least one of the basic variables will have an associated value of zero.

(c) *Starting from a degenerate current tableau, any pivot must be degenerate.* **False.** We know that if a tableau is degenerate, then at least one of the basic variables must have an associated value of zero. We also know that by definition a degenerate pivot is a pivot in which the associated solution and the basic objective function do not change between the starting and ending tableaus. However, we have seen that a pivot may have a starting degenerate tableau and an ending nondegenerate tableau. We may refer to Pivot 3 from Problem 2 as proof of a counter example:

$$
\begin{array}{llllll}
z - 3x_1 & & & & + 3s_4 & = 0 \\
2x_1 & & & +s_1 + s_2 & - 2s_4 & = 4 \\
- x_1 & & + x_3 & & + s_4 & = 0 \\
2x_1 & & & & +s_3 - s_4 & 4 \\
- x_1 & +x_2 & & - s_2 & + s_4 & = 0
\end{array}
$$

$$
\begin{aligned}
z &= 0 \\
B &= \begin{pmatrix} 2 & 3 & 4 & 6 \end{pmatrix} \\
x &= \begin{pmatrix} 0 & 0 & 0 & 4 & 0 & 4 & 0 \end{pmatrix} \\
\text{Entering Variable} &: x_1 \\
\text{Min Ratio Test} &: \begin{matrix} s_1 & \frac{4}{2} \\ s_3 & \frac{4}{2} \end{matrix} \\
\text{Leaving Variable} &: s_1 (\text{Lower Subscript})
\end{aligned}
$$

Pivot 3:

$$
\begin{array}{llllll}
z & & & +1.5s_1 + 1.5s_2 & & = 6 \\
1x_1 & & & +0.5s_1 + 0.5s_2 & - 1s_4 & = 2 \\
 & & 1x_3 & +0.5s_1 + 0.5s_2 & & = 2 \\
 & & & -1s_1 - 1s_2 & 1s_3 + 1s_4 & = 0 \\
 & 1x_2 & & +0.5s_1 - 0.5s_2 & & = 2
\end{array}
$$

We see here that the starting tableau was degenerate because the basic variables x_2 and x_3 have the associated values of 0. However when we pivot on the entering variable x_1, we see that the pivot is not degenerate since the objective function value increases from 0 to 6. Therefore, we may conclude that any pivot starting from a degenerate tableau may not necessarily be degenerate.

(d) *If the new tableau is degenerate, the pivot must have been degenerate.* **False.** We know that if the new tableau is degenerate than at least one of the values associated with a basic variable must be zero. We also know that with a degenerate pivot the objective function value and associated solution do not change between starting and and tableaus. Since we have previously shown that a degenerate pivot must start from a degenerate tableau in part a.), we know that the given statement is logically equivalent to the following: "If the new tableau is degenerate, then the starting tableau must have been degenerate". However, we have seen in our coursework that this is not the case. The following example is taken from the degeneracy section of our course book:

Current Tableau:

$$\begin{array}{llll} z - 2x_1 & -x2 & & = 0 \\ x_1 & -x_2 + x_3 & & = 1 \\ x_1 & & +x_4 & = 1 \\ & x_2 & x_5 & = 1 \end{array}$$

$$z = 0$$
$$B = \begin{pmatrix} 3 & 4 & 5 \end{pmatrix}$$
$$x^T = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$
$$\text{Entering Variable} : x_1$$
$$\text{Min Ratio Test} : \begin{matrix} x_3 & \frac{1}{1} \\ x_4 & \frac{1}{1} \end{matrix}$$
$$\text{Leaving Variable} : x_3 (\text{Lower Subscript})$$

Pivot 1:

$$\begin{array}{llll} z & -3x_2 + 2x_3 & & = 2 \\ x_1 & -x_2 + x_3 & & = 1 \\ & x_2 - x_3 & +x_4 & = 0 \\ & x_2 & + x_5 & = 1 \end{array}$$

$$z = 2$$
$$B = \begin{pmatrix} 1 & 4 & 5 \end{pmatrix}$$
$$x^T = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$
$$\text{Entering Variable} : x_2$$
$$\text{Min Ratio Test} : \begin{matrix} x_4 & \frac{0}{1} \\ x_5 & \frac{1}{1} \end{matrix}$$
$$\text{Leaving Variable} : x_4$$

We see that the starting tableau is not degenerate since the basic Note that the resulting tableau is degenerate since the basic variables all have associated values of 1. We also see that the resulting tableau is degenerate since x_4 has an associated value of 0. However, we note that the pivot was not degenerate in that both the associated solution and the objective function values changed. Therefore, if a new tableau is degenerate, it is not the case that the pivot must have been degenerate.