Advanced Calculus II
Unit 7.1: Exercises: 7.1.5, 7.1.8, 7.1.12
Unit 7.2: 7.2.2, 7.2.3, 7.2.5e, 7.2.5g, 7.2.6, 7.2.7

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7.1.5

Determine all values of p and q for which the following series converges: $\sum_{k=2}^{\infty} \frac{1}{k^q(\ln k)^p}$.

Case I, p >1 Let $p' > 0$ be a small number such that $p - p' > 1$. Let $a_k = \frac{1}{k^p(\ln k)^q}$ and $b_k = \frac{1}{k^{p-p'}}$. Then, by the limit comparison test,

$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim k \to \infty \frac{\frac{1}{k^p(\ln k)^q}}{\frac{1}{k^{p-p'}}} = \lim_{k\to\infty} \frac{(\ln k)^{-q}}{k^{p'}} = 0$, as demonstrated in example 7.1.6.

Since $p - p' > 1$, the series $\sum_{k=2}^{\infty} a_k = \sum_{k\to\infty} \frac{1}{k^{p-p'}}$ converges by the p-series, the series $\sum_{k=2}^{\infty} b_k = \sum_{k=2}^{\infty} \frac{1}{k^p(\ln k)^q}$ converges by the limit comparison test for all $p > 1$.

Case II, p = 1 Let $f(x) = \frac{1}{x(\ln x)^q}$. Then

$f'(x) = -\frac{(lnx)^q + q(\ln x)^{q-1}}{x^2(\ln x)^{q+1}} < 0$, therefore $f(x)$ is a monotone decreasing function. By the integral test.

$\int_2^{\infty} f(x)dx = \int_2^{\infty} \frac{1}{x(\ln x)^q} = \int_{\ln 2}^{\infty} \frac{1}{u^q}$, which is a p-series.

Therefore, for $p = 1$, t he series $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^q}$ converges if $q > 1$ and diverges if $q \leq 1$.

Case III, p <1 Let $p' > 0$ be a small number such that $p + p' < 1$. Let $a_k = \frac{1}{k^p(\ln k)^q}$ and $b_k = \frac{1}{k^{p+p'}}$. Then, by the limit comparison test,

$\lim\limits_{k\to\infty} \frac{b_k}{a_k} = \lim\limits_{k\to\infty} \frac{\frac{1}{k^{p+p'}}}{\frac{1}{k^p(\ln k)^q}} = \lim\limits_{k\to\infty} \frac{(\ln k)^q}{k^{p'}} = 0$, as demonstrated in example 7.1.6.

Since $p + p' < 1$, the series $\sum\limits_{k=2}^{\infty} b_k = \sum\limits_{k=2}^{\infty} \frac{1}{k^{p+p'}}$ diverges by the p-series and $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty} \frac{1}{k^p(\ln k)^q}$ diverges by the limit comparison test.

7.1.8

If $\sum a_k$ converges and $\sum b_k = \infty$, prove that $\sum(a_k + b_k) = \infty$.

First, we will show that $\sum\limits_{k=1}^{\infty} (a_k - b_k) = \alpha - \beta$ if $\sum_{k=1}^{\infty} a_k = \alpha$ and $\sum\limits_{k=1}^{\infty} b_k = \beta$.

For all $n \in \mathbb{N}$, let $s_n = \sum\limits_{k=1}^{n}$ and $t_n = \sum\limits_{k=1}^{n} -b_k$.

Since the series a_k and b_k converge to α and β respectively, $\lim s_n = \alpha$ and $\lim t_n = -\beta$. Therefore, by the algebra of limits, $\lim(s_n - t_n) = \alpha - \beta$.

Yet, $s_n - t_n = \sum\limits_{k=1}^{\infty} a_k - \sum\limits_{k=1}^{n} b_k = \sum\limits_{k=1}^{n} (a_k - b_k)$.

Therefore, $s_n - t_n$ is the nth partial sum of the series $\sum(a_k - b_k)$. Since the sequence $\{s_n - t_n\} = \alpha - \beta$ and $\sum\limits_{k=1}^{\infty} (a_k - b_k) = \alpha - \beta$.

Now, assume for the sake of contradiction that if $\sum a_k$ converges and $\sum b_k = \infty$, prove that $\sum(a_k + b_k)$ converges.

Since $\sum a_k$ converges, we know from above that if $\sum(a_k + b_k)$ converges, then their difference must also converge.

$\sum((a_k + b_k) - a_k) = \sum(a_k + b_k - a_k) = \sum b_k$, which is a contradiction since $\sum b_k$ diverges by assumption.

Therefore, if $\sum a_k$ converges and $\sum b_k = \infty$, prove that $\sum(a_k + b_k) = \infty$.

7.1.12

Suppose that the series $\sum a_k$ of positive real numbers converges by virtue of the root or ratio test. Show that the series $\sum_{k=1}^{\infty} k^n a_k$ converges for all $n \in \mathbb{N}$.

Since $\sum a_k$ converges by the root test $\limsup_{k\to\infty} \sqrt[k]{a_k} = \alpha_{a_k} < 1$

For all $n \in \mathbb{N}, \limsup_{k\to\infty} \sqrt[k]{k^n} = \limsup_{k\to\infty} (\frac{k+1}{k})^n = 1$, which implies that $b_k = k^n$ converges.

$b_k = k^n$ is a nonnegative series and a_k is a nonnegative series then $b_k a_k = k^n a_k$ is a nonnegative series. Thus, we can use the root test on the series $a_k b_k$.

$\limsup_{k\to\infty} \sqrt[k]{b_k a_k} = \limsup_{k\to\infty} \sqrt[k]{b_k} \sqrt[k]{a_k} = \limsup_{k\to\infty} \sqrt[k]{b_k} \limsup_{k\to\infty} \sqrt[k]{a_k}$, by the algebra of limits.

$= \limsup_{k\to\infty} \sqrt[k]{k^n} \limsup_{k\to\infty} \sqrt[k]{a_k} = \limsup_{k\to\infty} (\frac{k+1}{k})^n \limsup_{k\to\infty} \sqrt[k]{a_k} = 1 * \alpha_{a_k} = \alpha_{a_k} < 1$.

Thus, the series $\sum_{k=1}^{\infty} k^n a_k$ converges for all $n \in \mathbb{N}$ if the series $\sum a_k$ of positive real numbers converges by virtue of the root or ratio test.

7.2.2

Show by example that the hypothesis of the Alternating Series Test cannot be replaced by $b_k \geq 0$ and $\lim_{k\to\infty} b_k = 0$.

$$b_k = \begin{cases} \frac{1}{n+1}, & \text{if } k = 2n+1 \\ 0, & \text{if } k = 2n \end{cases}$$

Then the even partial sums of the series $\sum_{k=1}^{\infty} (-1)^{k+1} b_k$ is $s_{2n} = \sum_{k=1}^{n} \frac{1}{k}$ which diverges, but the odd partial sums $s_n = \sum_{k=1}^{n} 0$ converges. That is why the theorem requires that the series be decreasing.

7.2.3

If $\sum a_k$ *converges, does* $\sum a_k^2$ *always converge?*

Since $\sum a_k$ converges, we know by corollary 2.7.5 that $\lim_{k\to\infty} a_k = 0$

Let $b_n = a_n^2$. Then,

$\lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} \frac{a_n^2}{a_n} = \lim_{n\to\infty} a_n = 0.$

Therefore, $\sum a_k$ converges, $\sum a_k^2$ converges.

7.2.5e

Test the following series for convergence: $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^k}{(k+1)^{k+1}}$.

Let $b_k = \frac{k^k}{(k+1)^{k+1}}$.

$\lim_{k\to\infty} b_k = \lim_{k\to\infty} \frac{k^k}{(k+1)^{k+1}} = \lim_{k\to\infty} (k+1)^{-k-1}(k^k) = 0$

Let $f(x) = \frac{x^x}{(x+1)^{x+1}}$.

$f'(x) = x^x(x+1)^{-x-1}(\ln(x) - \ln(x+1)) \leq 0$, therefore b_k is decreasing.

Sinceb_k is decreasing and $\lim_{k\to\infty} b_k = 0$, we can apply the alternating series test.

By the alternating series test, $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^k}{(k+1)^{k+1}}$ converges.

7.2.5g

Test the following series for convergence: $\sum_{k=1}^{\infty} \frac{\sin kt}{k^p}, t \in \mathbb{R}, p > 0.$

Let $b_k = \frac{1}{k^p}$.

$g(x) = \frac{1}{x^p}$ and $g'(x) = \frac{-p}{x^{p-1}} \leq 0 \forall x \in \mathbb{R}$ and $p > 0$. Therefore, b_k is a decreasing sequence.

$\lim_{k\to\infty} \frac{1}{k^p} = 0, \forall p > 0.$

Therefore, we can apply the trignometric series theorem.

Thus, $\sum_{k=1}^{\infty} b_k * \sin(kt) = \sum_{k=1}^{\infty} \frac{\sin kt}{k^p}$ converges for all $t \in \mathbb{R}, p > 0$.

7.2.6

Given that $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{12}$*, determine how large* $n \in \mathbb{N}$ *must be chosen so that* $|\frac{\pi^2}{12} - s_n| < 10^{-4}$*, where* s_n *is the* n*th partial sum of the series.*

Let $b_k = \frac{1}{k^2}$.

$\lim_{k\to\infty} b_k = \lim_{k\to\infty} \frac{1}{k^2} = 0$

Let $f(x) = \frac{1}{x^2}$.

$f'(x) = \frac{-2}{x^3} < 0$.

Therefore b_k is a positive, monotone decreasing function and b_k satisfies the conditions of the alternating series test.

By the alternating series theorem, if $s = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$, then $s_n \leq s \leq s_{n+1}$ when n is even and $s_{n+1} \leq s \leq s_n$ when n is odd.

For even n, this implies that $0 \leq s - s_n \leq s_{n+1} - s_n$ and $|s - s_n| \leq |s_{n+1} - s_n|$.

For odd n, this implies that $s_{n+1} - s_n \leq s - s_n \leq 0$ and $|s - s_n| \leq |s_{n+1} - s_n|$.

Therefore, we know that for all n, $|s - s_n| \leq |s_{n+1} - s_n|$.

Yet, by the alternating series theorem, we know that $|s_{n+1} - s_n| = b_{n+1}$.

Thus, $|s - s_n| \leq b_{n+1}$ for all $n \in \mathbb{N}$.

Therefore $|s - s_n| = |\sum \frac{(-1)^{k+1}}{k^2} - s_n| = |\frac{\pi^2}{12} - s_n| < b_{k+1}$ where

$b_{k+1} = \frac{1}{(k+1)^2} < 10^{-4}$

Which implies $(k+1)^2 > 10^4$ and $k + 1 > 100$.

Therefore, when $n = 100$, $|\frac{\pi^2}{12} - S_n| < 10^{-4}$.

7.2.7

If p *and* q *are real numbers, show that* $\sum_{k=2}^{\infty} (-1)^k \frac{(\ln k)^p}{k^q}$ *converges.*

$p > 0$, then let $a_k = \frac{(\ln k)^p}{k^q}$. Then $a_k \geq 0$.

Let $f(x) = \frac{(lnx)^p}{x^q}$. Then $f'(x) = (\frac{p}{x \ln x} - \frac{q}{x})x^{-q}(\ln x)^p$ which is negative when $q > \max\{1, e^{\frac{p}{q}}\}$. Thus a_k is eventually monotone decreasing.

$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{(\ln k)^p}{k^q} = 0$, as previously demonstrated.

By the alternating series test, since a_k is monotone decreasing and $\lim_{k\to\infty} = 0$, we know that the series $\sum_{k=2}^{\infty} (-1)^k \frac{(\ln k)^p}{k^q}$ converges when $q > 0$ for any positve $p \in \mathbb{R}$.