Modern Algebra
Homework 5
Chapter 6a
Read Problems 6.15, 6.16
Do 6.5, 6.7, 6.8, 6.9, 6.11, 6.12, 6.13, 6.20, 6.28
Proof

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6.5

Find every subgroup for each of the following groups. Compute the order and index of each subgroup that you find

a.)V_4
$H_0 =< e >$ is the trivial subgroup of order 1 and index 4.
$H_1 =< h >$ is a subgroup of order 2 and index 2.
$H_2 =< v >$ is a subgroup of order 2 and index 2.
$H_3 =< h, v >$ is a subgroup of order 2 and index 2.
$H_4 =< h, v >$ is the group itself of order 4 and index 1.

b.)C_5
$H_0 =< e >$ is the trivial group of order 1 and index 5.
$H_1 =< a >$ is the group itself of order 5 and index 1.

c.)S_3
$H_0 =< e >$ is the trivial subgroup of order 1 and index 6.
$H_1 =< f >$ is a subgroup of order 2 and index 3.
$H_2 =< fr >$ is a subgroup of order 2 and index 3.
$H_3 =< rf >$ is a subgroup of order 2 and index 3.

$H_4 =< r >$ is a subgroup of order 3 and index 2.
$H_5 =< r, f >$ is the group itself of order 6 and idex 1.

d.)C_8
$H_0 =< e >$ is the trivial subgroup of order 1 and index 8.
$H_1 =< a^4 >$ is a subgroup of order 2 and index 4.
$H_2 =< a^2 >$ is a subgroup of order 4 and index 2.
$H_3 =< a >$ is the group itself of order 8 and index 1.

e.)D_4
$H_0 =< e >$ is the trivial subgroup of order 1 and index 8.
$H_1 =< r^2 >$ is a subgroup of order 2 and index 4.
$H_2 =< f >$is a subgroup of order 2 and index 4.
$H_3 =< fr >$ is a subgroup of order 2 and index 4.
$H_4 =< r^2 f >$ is a subgroup of order 2 and index 4.
$H_5 =< rf >$ is a subgroup of order 2 and index 4.
$H_6 =< r >$ is a subgroup of order 4 and index 2.
$H_7 =< r^2, f >$ is a subgroup of order 4 and index 2.
$H_8 =< r^2, fr >$ is a subgroup of order 4 and index 2.
$H_9 =< r, f >$ is the group itself.

f.)$C_3 \times C_3$
$H_0 =<< e, e >>$ is the trivial subgroup of order 1 and index 9.
$H_1 =<< a, e >>$ is a subgroup of order 3 and index 3.
$H_2 =<< e, a >>$ is a subgroup of order 3 and index 3.
$H_3 =<< a, a >>$ is a subgroup of order 3 and index 3.
$H_4 =<< a^2, a >>$ is a subgroup of order 3 and index 3.
$H_5 =<< a, e >, < e, a >>$ is the group itself with order 9 and index 1.

6.7

a.)*If e is the identity element, then what is* $< e >$*?*
$< e >$ is the trivial subgroup. It is also the cyclic group C_1.

b.)*If a is in the subgroup* $< b, c >$*, then what is* $< a, b, c >$*?*
If a is in the subgroup, then $< a, b, c >=< b, c >$.

c.)*If a is not in the subgroup* $< b, c >$*, and that subgroup's index is 2, then*

what is $< a, b, c >$*?*
$< a, b, c >$ is the partition of the subgroup.

d.)*If a is not in the subgroup* $< b, c >$*, and that subgroup's order is 28, then what do you know about the order of* $< a, b, c >$*?* $| < a, b, c > | = n - 28$.

6.8

Consider a cyclic group C_n*. If* m *is a number that divides* n*, then describe all subgroups* C_n *has of order* m*.*
If m divides n and C_n is a cyclic group, then all subgroups of C_n with order m take on the form $H =< a^{\frac{n}{m}} >$.

6.9

If a is the permutation that interchanges the numbers 1 *and* 2*, but leaves all other numbers alone, then what is* $[S_n :< a >]$*?*
The index of $< a >= \frac{|S_n|}{|<a>|} = \frac{n!}{2}$ because the order of a permuation of the numbers $(1\ \ 2)$ is 2 for any group S_n.

6.11

A particular special case of Lagrange's Theorem bears highlighting on its own. As we learned in Chapter 5, any element g *in a group can be used to create a cyclic subgroup called the orbit of* g*. We write* $|g|$ *as shorthand for the size of that orbit (i.e.,*$|g| = | < g > |$*), and call it the order of* g*.*
Explain why the order $|g|$ *of any element must divide the order*$|G|$ *of the group.*

Let $g \in G$ have finite order such that $|a| =$ k. Then $| < g > | = k$ where $< g >$ is the smallest subgroup containing g. This implies that the order of a subgroup generated by element g is equal to the order of element g. By Lagrange's Theorem, the order of a subgroup $< g >$ must divide the order of the group G. Since $| < g > | = |g|$, we know that the order of any element g must also divide the order of the group.

6.12

a.)*How many subgroups does G have when $|G|$ is prime?*
It has 2 subgroups: the trivial group and the non-proper subgroup.

b.)*What is the orbit of an element in such a group G?*
The orbit of such an element in group G will include all elements in group G.

c.)*How many groups of order p are there, when p is prime?*
There is only one group of order p if p is prime.

6.13

a.)*What subgroup of A_4 is generated by the following two permutations? (1 2)(3 4) and (1 3)(2 4)*
First, we know that (1 2)(3 4)(1 3)(2 4) = (1 4)(2 3).

$|(1\ 4)(2\ 3)| = lcm(2, 2) = 2$. Therefore, the subgroup has order 2. There is only one subgroup of A_4 with order 2 and that is subgroup that is isomorphic to the cyclic group C_2.

b.)*What subgroup of S_4 is generated by the following two permutations? (1 2 3)(4) and (1)(2 3 4)*
First, we know that (1 2 3)(2 3 4) = (1 3)(2 4).

$|(1\ 3)(2\ 4)| = lcm(2, 2) = 2$. Therefore, the subgroup has order 2. Both subgroups of S_4 with order 2 are isomorphic to C_2, so this subgroup the subgroup that is isomorphic to the cyclic group C_2.

6.20

For each H and G given below, find all left cosets of H in G, then state the index $[G : H]$.

a.)$H =< 4 >, G = C_{20}$
The left cosets are: $1 + H, 2 + H, 3 + H, H$. Index is 4.

b.)$H =< 6 >, G = C_{15}$
The left cosets are: $1 + H, 2 + H, H$. Index is 3.

c.)$H = f, G = D_4$
The left cosets are: rH, r^2H, r^3H, H. Index is 4.

D.)$H = A_4, G = S_4$
The left cosets are: $(1\ 2)H, H$. Since the order of A_4 is $\frac{1}{2}$ the order of S_4, the index is 2.

6.28

For each of the following questions, either find a group that answers the question in the affirmative or give a clear explanation of why the answer to the question is negative.

a.)*Is there a non-cyclic group all of whose proper subgroups are cyclic?*
Yes, the unitary group $U_8 = \{1, 3, 5, 7\}$ where the binary operation is multiplication modulo 8 is non-cyclic, but all of its proper subgroups are cyclic.

b.)*Is there a non-abelian group all of whose proper subgroups are abelian?*
Yes. An example would be the group S_3, which is non-abelian, but its 5 proper subgroups are all Abelian.

Proof

Prove that every subgroup of a cyclic group is cyclic. (Do not assume that G is finite).

Let G denote the cyclic group generated by element a and H be any subgroup of G.

If H is the trivial subgroup $< e >$, then H is cyclic because the trivial group is isomorphic to C_1.

If H is the non-proper subgroup, then H is cyclic because $H = G$ and G is cyclic.

If $H \neq < e >$ and H is not the non-proper subgroup, then $a^n \in H$ for some $n \in \mathbb{Z}$.

Let $m = min\{integers|a^m \in H$.

Let $b \in H$ where b is arbitrary.

Since $H < G$ and G is cyclic, $b = a^n$ for some $n \in \mathbb{Z}$.

Since $n \in \mathbb{Z}$ there exists $q, r \in mathbbZ$ such that $n = mq + r$, where $0 \leq r < m$.

Therefore $a^n = a^{mq+r} = (a^m)^q a^r$ and $a^r = (a^m)^{-q} a^n$.

We know that since $a^m \in H$ and H is a group that the inverse $(a^m)^{-1} \in H$.

Since $a^m \in H$ and $(a^m)^{-1} \in H$ and H is a group, $a^m * (a^m)^{-1} = a^r \in H$.

However, since we let$m = min\{integers|a^m \in H$ and $0 \leq r<m$, we know that $r = 0$.

Therefore, we must have that $n = qm$ and $b = a^n = (a^m)^q$.

This implies that any arbitrary element $b = a^n$ of H is generated by a^m. Therefore, $H =< a^m >$, which is cyclic.

Therefore, subgroup of a cyclic group is also cyclic.