Modern Algebra
Homework 6b
Chapter 6
Read 6.15, 6.16
Complete 6.17, 6.18, 6.22, 6.23, 6.24, 6.29
Proof

Megan Bryant

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6.17

Consider the group $\mathbb{Z}$, which is a group under the operation of ordinary addition.

a.)*If n is in $\mathbb{Z}$, then is $\langle n \rangle$ a subgroup of $\mathbb{Z}$? What does it contain?*

Since $\mathbb{Z}$ is infinite, we can have subgroups of any possible order. Thus $\langle n \rangle < Z$ and $< n >=$ all multiples of n.

b.)*For which $n \in \mathbb{Z}$ is $\langle n \rangle = \mathbb{Z}$?*

If $n = 1, -1$, then $\langle n \rangle = \mathbb{Z}$.

c.)*If n and m are both in $\mathbb{Z}$, then when is $\langle n \rangle < \langle m \rangle$?*

$\langle n \rangle < \langle m \rangle$ when $|\langle m \rangle|$ divides $|\langle n \rangle|$.

d.)*Can $\langle n \rangle$ itself also have subgroups?*

Yes, because $\langle n \rangle$ is a group, it has at least the trivial subgroup and the non-proper subgroup (these might be the same if $\langle n \rangle$ is the trivial group).

e.)*What subgroup of $\mathbb{Z}$ is generated by 2 and 5?*

The subgroup generated is $\langle 1 \rangle = \mathbb{Z}$ since 2 and 5 are distinct primes they can be used to generate the entire group.

f.) *What subgroup of* $\mathbb{Z}$ *is generated by* 4 *and* 6*?*

The subgroup generated is $\langle 2 \rangle$ which is the even numbers, since odd numbers cannot be generated from two even numbers when addition is the binary opperation.

6.1.8

a.) *What is the one coset of* $\langle 2 \rangle$ *in* $\mathbb{Z}$*?*

$1 + \langle 2 \rangle$ is the only coset (not including the trivial subgroup and non-proper subgroup).

b.)*How many cosets does* $\langle 3 \rangle$ *have in* $\mathbb{Z}$*, and what are they?*

Not including the trivial subgroup and non-proper subgroup, $\langle 3 \rangle$ has 2 cosets $\in \mathbb{Z}$. They are $1 + \langle 3 \rangle$ and $2 + \langle 3 \rangle$.

c.)*Is* $\langle n \rangle$ *a normal subgroup of* $\mathbb{Z}$*?*

Yes, all subgroups of $\mathbb{Z}$ are normal because addition is commutative.

6.22

a.) *Why are there no lines among any of the subgroups* $\langle f \rangle$, $\langle rf \rangle$, $\langle r^2 f \rangle$, *and* $\langle r \rangle$ *in the Hasse diagram for* S_3*?*

There are no lines connecting them because they are not subgroups of eachother.

b.) *What is the smallest subgroup of* S_3 *containing both* f *and* rf*?*

The smallest subgroup containg both f and rf is the non-proper subgroup. This is evident from the Hasse diagram, since there are no lines containing $\langle f \rangle$ and $\langle rf \rangle$.

c.) *What is the smallest subgroup of* C_5 *with more than one element in it?*

The only subgroups of C_5 are the trivial and non-proper subgroups. Therefore, the smallest subgroup of C_5 with more than one element is $\langle a \rangle$.

d.)*Label each line in each of the above two Hasse diagrams with the index of the smaller subgroup in the larger. For example, you would label the lower-leftmost line in the* S_3 *diagram with* 2 *because* $[\langle f \rangle \colon \{e\}] = 2$.

6.23

Make a Hasse diagram for each of the following groups: $V_4, C_5, S_3, C_8, D_4,$ $C_3 \times C_3$.

a.)V_4:

b.)C_5:

c.)S_3:

d.)C_8:

e.)D_4:

f.)$C_3 \times C_3$:

6.24

Make a Hasse diagram for C_{24}.

6.29

For each of the following questions, either find a group that answers the question in the affirmative or give a clear explanation of why the answer to the question is negative.

a.)*Is there a group of order* 8 *with a subgroup whose cosets partition the group into two different cosets (each coset therefore containing four elements)?*

Yes, all of the groups of order 8 have subgroups of order 4 which will partition the group into two different cosets.

b.)*Is there a group of order* 8 *with a subgroup whose cosets partition the group into eight different cosets (each element in its own coset)?*

Yes, every group contains the trivial subgroup which has index 8, therefore each element is partitioned into its own coset.

c.)*Is there a group of order* 8 *with a subgroup whose cosets partition the group into just one big cluster (every element in one coset)?*

Yes, every group has the non-proper subgroup, whose cosets partition the group into a one cluster.

d.)*Is there a group of order* 30 *with a subgroup whose cosets partition the group into one big cluster (every element in one coset)?*

Yes, every group has the non-proper subgroup, whose cosets partition the group into a one cluster.

e.)*Is there an abelian group with a subgroup whose left and right cosets partition the group differently?*

No. Every subgroup of an abelian group is normal. Proof: If H is a subgroup of the Abelian group G and $g \in H, h \in H$, then $ghg^{-1} = hgg^{-1} = he = h \in H$.

Proof

a.)*If $\mathbb{H}$ is a collection of subgroups of G, then the intersection $\cap_{H \in \mathbb{H}} H$ is a subgroup of G.*

All subgroups are groups, therefore $e \in \mathbb{H}$, where e is the identity element. Therefore, $e \in \cap_{H \in \mathbb{H}} H$.

Let $a \in \cap_{H \in \mathbb{H}} H$, then $a \in \mathbb{H}$. Since each subgroup in $\mathbb{H}$ is a group, $a^{-1} \in \mathbb{H}$. Therefore, $a^{-1} \in \cap_{H \in \mathbb{H}} H$.

Let $a, b \in \cap_{H \in H} H$, then $a, b \in$ all subgroups of $\mathbb{H}$. Since the subgroups in the collection H are groups, they are closed under the group operation. Therefore $ab \in$ all subgroups of $\mathbb{H}$. Thus, $ab \in \cap_{H \in H} H$

Therefore, $\cap_{H \in \mathbb{H}} H$ is a subgroup of G if $\mathbb{H}$ is a collection of subgroups of G.

b.)*If $S \subset G$, then $\langle S \rangle$ is the intersection of all subgroups containing S. (Hint: One way is to prove that $A = B$ is to show $A \subset B$ and $B \subset A$).*

Let $A = \{H < G | S \subset H\}$.

Since $S \subset G$, $G \in A$ and $A \neq \emptyset$ and $S' = \cap_{H \in A}$ is well defined.

By definition, the group generated by S is the smallest subgroup which contains S. Therefore $\langle S \rangle \in A$ and, therefore $S' \subset \langle S \rangle$.

Suppose $\langle S \rangle \not\subset S'$, then $\langle S \rangle \cap S'$ contradicts the fact that $\langle S \rangle$ is the smallest group which contains S.

Therefore, $\langle S \rangle \subset S'$.

Thus, $\langle S \rangle \subset S'$ and $S' \subset \langle S \rangle$ which implies $\langle S \rangle = S' = \cap_{H \in A}$.

c.)*Prove that if $x \in H$, then $xH = H$. What is the interpretation of this statement in terms of the Cayley diagram?*

Since H is closed under the group operation $xH \subseteq H$

Since $x \in H$, we have $x^{-1} \in H$

For any $h \in H$, we know that $x^{-1}h \in H$.

Therefore, $h = eh = (xx^{-1})h = x(x^{-1}h) \in xH$

Thus $h \in xH$ and $H \subseteq aH$.

Therefore, if $x \in H$, then $xH = H$.

In terms of the Cayley diagram, this means that if you start at any node in the subgroup H and follow the actions in H, you will generate the same subgroup.

d.)*Prove that if $b \in aH$, then $aH = bH$. (Use the definition of a coset: $aH = \{ah : h \in H\}$)*

If $b \in aH = \{ah : h \in H\}$, then $b = ah$ for some $h \in H$.

$b = ah$

$a^{-1}b = a^{-1}ah = eh = h$

Therefore $h = a^{-1}b \in H$.

Take any $bh \in bH = \{bh : h \in H\}$

$bh = (aa^{-1})bh = a(a^{-1}b)h \in aH$ because $(a^{-1}b)h \in H$ since both $(a^{-1}b)$ and h are in H and H is closed under its binary operation.

Therefore $bH \subseteq aH$.

Take and$ah \in aH$.

$ah = bb^{-1}ah = b(a^{-1}b)^{-1}h \in bH$ since $(a^-1b)^{-1}h \in H$ because H contains the inverse of all its elements and both $(a^{-1}b)$ and h are in H and H is closed under its binary operation.

Therefore, $aH \subseteq bH$.

Thus if $b \in aH$, then $aH = bH$.