LINEAR PROGRAMMING

Homework 6

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1. *Suppose A is an m-by-n matrix and c is a column n-vector. In this question, you will show that for any choice of A and c, exactly one of the following two systems of inequalities has a solution.*

$$(*) Ax = 0, x \geq 0, \text{and } c^T x > 0$$
$$(**) A^T y \geq c$$

(a) *Explain why it cannot be the case that both (*) and (**) have a solution.*

Let us suppose that we have vectors x^* and y^{**} such that x^* is a feasible solution for $(*)$ and y^{**} is a feasible solution for $(**)$. Then we know that for any index $i = 1, \ldots, n$, we have

$$x_i^* \geq 0 \ \& \ (A^T y^{**})_i > c_i^*.$$

From this, we can conclude that

$$(A^T y^{**})_i x_i^* \geq c_1^* x_i^*$$

which implies that

$$\sum_{i=1}^{n} (A^T y^{**})_i x_i^* \geq \sum_{i=1}^{n} c_i x_i^*.$$

Given these inequalities, we can conclude the following

$$c^T x^* \leq (A^T y^{**})^T x^* = ((y^{**})^T A) x^* = (y^{**})^T (Ax^*) = 0, \text{ since, } Ax^* = 0 \text{ by } (*).$$

However, this is a violation of the constraints of (*), which state that $c^T x$ must be strictly positive. Therefore, we can conclude that it cannot be the case that both x^* and y^{**} are feasible solutions to $(*)$ and $(**)$, respectively.

(b) *Suppose the system (*) has no solution. You will show that this means that (**) must have a solution.*

i. *Suppose (*) has no solution. Explain what this means about the optimal solution of the following linear program where $b = 0$.*

$$\begin{aligned} & \max\ c^T x \\ \text{(P)}\ & \text{s.t. } Ax = b \\ & x \geq 0 \end{aligned}$$

Given that $b = 0$, the given linear program may be rewritten as follows

$$\begin{aligned} &\max\ c^T x \\ (\text{P})\ &\text{s.t. } Ax = 0 \\ &x \geq 0 \end{aligned}$$

We now note that our linear program consists incorporates all elements of $(*)$ except for $c^T x > 0$. Furthermore, given that $(*)$ has no solution, we know that a feasible optimal solution for our given linear program exists only if it meets the additional constraint $c^T x \leq 0$. Therefore, we may further rewrite our linear program as follows

$$\begin{aligned} &\max\ c^T x \\ (\text{P})\ &\text{s.t. } Ax = 0 \\ &c^T x \leq 0 \\ &y \text{ is free} \end{aligned}$$

Given this equivalent linear program and the fact that we are seeking to maximize $c^T x$, we can conclude that the optimal objective value is 0. This is the only possible optimal objective value given that $(*)$ has no solution.

ii. *Explain why the system (**) must have a solution by considering the dual of (P) and explaining what you know about it (using part (b)i).*
Given the above primal linear program (P), we can derive the following dual linear program

$$\begin{aligned} &\min\ b^T y \\ (\text{D})\ &\text{s.t. } A^T y > c \\ &x \geq 0 \end{aligned}$$

However, given that $b = 0$, we can simplify (D) to the following program

$$\begin{aligned} &\min\ 0 \\ (\text{D})\ &\text{s.t. } A^T y > c \\ &y \text{ is free} \end{aligned}$$

The Strong Duality Theorem tells us that if the primal problem (P) has an optimal objective value, then the dual (D) must also have an optimal objective value and their objective values must equal. We have previously shown that (P) has an optimal objective value of 0. Then, by the Strong Duality Theorem, we know that (D) must also have an optimal objective value of 0.
Now, we must consider the implications for (**). We should note that our primal (P) (and thus our dual (D)) was found to have an optimal solution on the condition that $(*)$ has no solution. In addition, we know from part (a) that it is impossible for both (*) and (**) to have a solution. Given this, we can conclude that since the dual linear program (D) has a solution and that program has only one constraint, $A^T y \geq c$ (which is the entirety of (**)), (**) must have a solution if (*) does not have a solution.

2. *A car factory makes various kinds of cars. Each kind needs a certain number of hours per car to produce and yields a certain profit per car. The factory must operate for exactly 120 hours per week and must meet or exceed certain weekly production levels for each car. The different kinds of car, production times, profits (in dollars), and production levels are shown in the following table.*

kind of car	hours to produce	profit	production level
economy	1	200	10
standard	2	500	20
luxury	3	700	15

We can model the problem of maximizing the weekly profit using the following linear program:

$$\begin{aligned} \max\ & 200x_1 + 500x_2 + 700x_3 \\ \text{s.t. } & x_1 + 2x_2 + 3x_3 = 120 \\ & x_1 - x_4 = 10 \\ & x_2 - x_5 = 20 \\ & x_3 - x_6 = 15 \\ & x_i \geq 0, i = 1, \dots, 6 \end{aligned}$$

Consider the basis $B = [1, 2, 3, 5]$.

(a) *What do the variables $x_1, x_2, x_3, x_4, x_5, x_6$ represent?*

x_1 : The total number of economy cars produced.
x_2 : The total number of standard cars produced.
x_3 : The total number of luxury cars produced.
x_4 : The number of economy cars produced in excess of demand.
x_5 : The number of standard cars produced in excess of demand.
x_6 : The number of luxury cars produced in excess of demand.

(b) *Calculate the basic solution x^* corresponding to basis B.*

We know that we can used the revised simplex method to caluclate a basic solution for a starting basis. First, we must find the appropriate matrices, where A_B is the basis matrix and b is the right hand side of the constraints.

$$A_B = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad A_B^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{-1}{2} & 0 & \frac{-3}{2} \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & -1 & \frac{-3}{2} \end{pmatrix} \quad b = \begin{pmatrix} 120 \\ 10 \\ 20 \\ 15 \end{pmatrix}$$

Now, we can use these matrices to calculate x_B^*.

$$x_B^* = A_B^{-1} \times b = \begin{pmatrix} 10 \\ 32.5 \\ 15 \\ 12.5 \end{pmatrix}.$$

We know that all nonbasic varianles are set to zero in a basic solution. Therefore, we arrive at the following solution for basis B:

$$x^* = \begin{pmatrix} 10 \\ 32.5 \\ 15 \\ 0 \\ 12.5 \\ 0 \end{pmatrix}$$

(c) *Verify that x^* is a nondegenerate basic feasible solution.* First, to verify that x^* is in fact a solution, we must verify that x^* satisifies all constraints.

$$\begin{aligned} 1(10) + 2(32.5) + 3(15) &= 120 \checkmark \\ 1(10) - 1(0) &= 10 \checkmark \\ 1(32.5) - 1(12.5) &= 20 \checkmark \\ 1(15) - 1(0) &= 15 \checkmark \end{aligned}$$

Therefore, x^* is a solution to the given linear program.

We know that x^* is a nondegenerate solution since the number of zeros is equal to the number of nonbasic variables (e.g. there are two zeroes).

We know that x^* is a feasible solution since all values are nonnegative which meets the feasibility constraints.

Therefore, x^* is a nondegenerate, basic feasible solution to the linear program for basis B.

(d) *By starting the revised simplex method at basis B, verify that x^* is optimal.*

The basis and the current value of the basic variables is

$$B = [1, 2, 3, 5], x_B^* = \begin{pmatrix} 10 \\ 32.5 \\ 15 \\ 12.5 \end{pmatrix}, c = \begin{pmatrix} 200 \\ 500 \\ 700 \\ 0 \\ 0 \\ 0 \end{pmatrix} A = \begin{pmatrix} 1 & 2 & 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}.$$

Now we must compute the vector y.

$$A_B^T y = c_B$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 200 \\ 500 \\ 700 \\ 0 \end{pmatrix}$$

By solving the augmented matrix, we solve for

$$y = \begin{pmatrix} 250 \\ -50 \\ 0 \\ -50 \end{pmatrix}$$

Now we will attempt to select an entering index k.

k	c_k		$A_k^T y$
4	0	$\leq$	50
6	0	$\leq$	50

We see that for each value of k the entering variable x_k does not have a positive reduced cost which means that we do not have a valid choice of entering variable and our solution x^* is optimal.

(e) *Write down the dual problem.*

Recall that our given primal linear program is as follows

$$\begin{aligned} \text{(P) max } & 200x_1 + 500x_2 + 700x_3 \\ \text{s.t. } & x_1 + 2x_2 + 3x_3 = 120 \\ & x_1 - x_4 = 10 \\ & x_2 - x_5 = 20 \\ & x_3 - x_6 = 15 \\ & x_i \geq 0, i = 1, \ldots, 6 \end{aligned}$$

Therefore our derived dual linear program would be

$$\begin{aligned} \text{(D) min } & 120y_1 + 10y_2 + 20y_3 + 15y_4 \\ \text{s.t. } & y_1 + y_2 \geq 200 \\ & 2y_1 + y_3 \geq 500 \\ & 3y_1 + y_4 \geq 700 \\ & -y_2 \geq 0 \\ & -y_3 \geq 0 \\ & -y_4 \geq 0 \\ & y_1 \text{ is free} \end{aligned}$$

This can be rewritten as

$$\begin{aligned} \text{(D) min } & 120y_1 + 10y_2 + 20y_3 + 15y_4 \\ \text{s.t. } & y_1 + y_2 \geq 200 \\ & 2y_1 + y_3 \geq 500 \\ & 3y_1 + y_4 \geq 700 \\ & y_2, y_3, y_4 \leq 0 \\ & y_1 \text{ is free} \end{aligned}$$

(f) *Write down the complementary slackness conditions that hold if and only if x^* is optimal and, hence, give another verification that x^* is optimal.*

Recall that

$$x^* = \begin{pmatrix} 10 \\ 32.5 \\ 15 \\ 0 \\ 12.5 \\ 0 \end{pmatrix}$$

We know that the following is the set of all complimentary slackness conditions:

$$y_1(x_1 + 2x_2 + 3x_3 - 120) = 0 \tag{1}$$
$$y_2(x_1 - x_4 - 10) = 0 \tag{2}$$
$$y_3(x_2 - x_5 - 20) = 0 \tag{3}$$
$$y_4(x_3 - x_6 - 15) = 0 \tag{4}$$
$$x_1(y_1 + y_2 - 200) = 0 \tag{5}$$
$$x_2(2y_1 + y_3 - 500) = 0 \tag{6}$$
$$x_3(3y_1 + y_4 - 700) = 0 \tag{7}$$
$$x_4(-y_2) = 0 \tag{8}$$
$$x_5(-y_3) = 0 \tag{9}$$
$$x_6(-y_4) = 0 \tag{10}$$

The complementary slackness conditions that must hold if x^* is optimal are the conditions associated with the basic variables. That is to say the following conditions must hold true

$$\begin{aligned} y_1 + y_2 - 200 &= 0, \text{ since } x_1^* \neq 0 \\ 2y_1 + y_3 - 500 &= 0, \text{ since } x_2^* \neq 0 \\ 3y_1 + y_4 - 700 &= 0, \text{ since } x_3^* \neq 0 \\ -y_3 &= 0, \text{ since } x_5^* \neq 0 \end{aligned}$$

Solving the above system of equations gives us the following result:

$$y^* = \begin{pmatrix} 250 \\ -50 \\ 0 \\ -50 \end{pmatrix}$$

This solution represents an optimal solution to the dual with a corresponding optimal objective value of \$28,750. That optimal objective value is also the optimal objective value associated with the basic feasible solution x^*. Therefore, x^* is an optimal solution by the complementary slackness conditions.

(g) *Write down an optimal solution of the dual problem.*

We know by **Corollary 16.9** that if the revised simplex method terminates with an optimal solution (as it did in part (d)), then the final vector y computed by the algorithm is optimal for the dual problem.

$$y^* = \begin{pmatrix} 250 \\ -50 \\ 0 \\ -50 \end{pmatrix}$$

We can verify this result using the Strong Dualty Theorem, which states that if a primal problem has an optimal objectevie value then the dual must also have an optimal objective value and that theses two objective values must equal. We know from (d) that x^* is an optimal solution giving an optimal objective value of \$28, 750.

Therefore, by the Strong Duality Theorem, $c^T x^* = \$28,750$ is also the optimal objective value for the dual problem (D).

$$b^T y^* = c^T x^* = 28750, \text{ where } b^T = \begin{pmatrix} 120 & 10 & 20 & 15 \end{pmatrix}$$

$$\begin{pmatrix} 120 & 10 & 20 & 15 \end{pmatrix} \times \begin{pmatrix} 250 \\ -50 \\ 0 \\ -50 \end{pmatrix} = 28750 \checkmark$$

We can also verify that y is feasibile for the dual problem (D) by ensuring that it meets the feasibility constraints, which it does.

(h) *Use your optimal dual solution to predict the price per hour that the factory would pay to increase the production time about 120 hours.*

We can use the corresponding dual variable for the constraint on production hours to determine what the factory would be willing to pay to increase production time.

Since our dual variable $y_1^* = 250$, we know that the factory would be willing to pay up to \$250 to increase production beyond 120 hours as they can infer that each additional hour of production would result in an increase of profits in the amount of \$250.

(i) *Suppose that the production level for luxury cars is increased from 15 to 30. Show that the current basis B no longer corresponds to a basic feasible solution.*

If we are to increase the production level of luxury cars from 15 to 30, we must update the corresponding primal constraint to reflect that change. This results in the following updated primal linear program

$$\begin{aligned} \max\ & 200x_1 + 500x_2 + 700x_3 \\ \text{s.t. }\ & x_1 + 2x_2 + 3x_3 = 120 \\ & x_1 - x_4 = 10 \\ & x_2 - x_5 = 20 \\ & x_3 - x_6 = 30 \\ & x_i \geq 0, i = 1, \dots, 6 \end{aligned}$$

We will once again employ the revised simplex method to calculate the basic solution for the starting basis $B = [1, 2, 3, 5]$ given the updated primal program.

$$A_B = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad A_B^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{-1}{2} & 0 & \frac{-3}{2} \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & -1 & \frac{-3}{2} \end{pmatrix} \quad b = \begin{pmatrix} 120 \\ 10 \\ 20 \\ 30 \end{pmatrix}$$

Now, we can use these matrices to calculate x_B^*.

$$x_B^* = A_B^{-1} \times b = \begin{pmatrix} 10 \\ 10 \\ 30 \\ -10 \end{pmatrix}.$$

We know that all nonbasic varianles are set to zero in a basic solution. Therefore, we arrive at the following solution for basis B:

$$x^* = \begin{pmatrix} 10 \\ 10 \\ 30 \\ 0 \\ -10 \\ 0 \end{pmatrix}$$

However, this solution x^* is infeasible for the updated primal linear program as $x_5^* = -10 \leq 0$ which violates the associated feasibility constraints. Therefore, if the production for luxury cars is increased from 15 to 30, the current basis B no longer corresponds to a basic feasible solution.

3. *Consider question 2 on Homework 4. Find some interesting information that duality provides that you can share with the production manager. To get the optimal dual variables, solve the problem in AMPL and type* ***display constraintname;*** *to get the dual value associated with the constraint* ***constraintname****.*

   AMPL Output

```
ampl: reset;model multiperiod.mod; option solver gurobi; solve;
Gurobi 5.6.3: optimal solution; objective 186898.3333
28 simplex iterations
ampl: display conservation; display storage_space; display limited_storage;
display x_init;display inventory_init; display inventory_exit;
conservation [*] :=
 1  2
 2  0.975
 3  1.025
 4  1.6
 5  1.65
 6  1.28333
 7  1.33333
 8  1.38333
 9  1.35
10  1.4
11  1.85
12  1.9
;

storage_space [*] :=
 0   0
 1   0
 2   0
 3   0
 4   0
 5   0
 6   0
 7   0
 8   0
 9   0
10  -0.4
11   0
12   0
;

limited_storage [*] :=
 0   0
 1   0
```

```
 2   0
 3  -0.525
 4   0
 5   0
 6   0
 7   0
 8   0
 9   0
10   0
11   0
;

x_init = 0.25

inventory_init = -2

inventory_exit = 1.95
```

We know that since we are dealing with a minimization problem that consists of $\leq$ constraints that we must multiply the supplied dual variable values to determine how much we would be willing to pay to adjust a constraint. In this problem, there is only one variable that is particularly feasible to change by price alone. That constraint is storagespace, which reflects the maximum amount of storage available. We see from the above AMPL output that we would be willing to pay up to \$0.40 to increase our storage capacity in period 10 (October).

We should also note that we would be willing to pay up to \$0.525 per unit to increase the shelf life in period 3 (March). This information might be valuable to the Research and Development team for product research.