Advanced Calculus II
Unit 7.3: 7.3.1a, 7.3.3a, 7.3.6b, 7.3.6f, 7.3.6h
Unit 7.4: 7.4.1b, 7.4.1c, 7.4.2b, 7.4.3, 7.4.6, 7.4.7

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7.3.1a

Prove the following: If $\lim\limits_{k\to\infty} k^p a_k = A$*, for some* $p > 1$*, then* $\sum a_k$ *converges absolutely.*

We know that $\lim\limits_{k\to\infty} k^p a_k = A, p > 1$ implies that $\lim\limits_{k\to\infty} k^p |a_k| = |A|$.

Therefore, $\exists N \in \mathbb{N}$ such that for all $k > N$, $k^p|a_k| - |A| < 1$.

Thus, $|a_k| < \frac{|A|+1}{k^p} = (|A|+1)\frac{1}{k^p}$, where (—A— + 1) is a number.

We know that for $p > 1$, $\frac{1}{k^p}$ is a convergent p-series.

Therefore, by the comparison test, we know that $\sum |a_k|$ converges and $\sum a_k$ converges absolutely.

7.3.3a

Prove that if $\sum a_k$ *converges and* $\sum b_k$ *converges absolutely, then* $\sum a_k b_k$ *converges.*

Since a_k converges, $\lim\limits_{k\to\infty} a_k = 0$ and a_k is bounded.

Therefore, for some $a_k \leq M, \forall k \in \mathbb{N}$ for some $M \in \mathbb{R}$.

Since b_k converges absolutely, by the Cauchy Criterion (theorem 2.7.3), give $\epsilon > 0, \exists N \in \mathbb{N}$ such that for all $n > m > N$,

$|S_n - S_m| = ||a_n| - |a_{n-1} + \cdots + |a_{m+1}|| \leq \frac{\epsilon}{M}$, where S_n is the n^{th} partial sum of $\sum_{k=1}^{\infty} |a_k|$.

For some $N \in \mathbb{N}$,

$|a_n b_n - a_m b_m| = a_n b_n + a_{n-1} b_{n-1} + \cdots + a_{m+1} b_{m+1}| \leq |a_n b_n| + |a_{n-1} b_{n-1}| + \cdots + |a_{m+1} b_{m+1}|$
$\leq M(|a_n| + |a_{n-1}| + \cdots + |a_{m+1} \leq M(||a_n| + |a_{n-1}| + \cdots + |a_{m+1}|| < M\frac{\epsilon}{M} = \epsilon$

Therefore, if a_k converges and b_k converges absolutely, $\sum a_k b_k$ converges.

7.3.6b

Test the following series for absolute and conditional convergence:
$\sum_{k=3}^{\infty} \frac{(-1)^k}{\sqrt{k}\ln(\ln k)}$.

First, we will apply the alternating series test.

Let $b_k = \frac{1}{\sqrt{k}\ln(\ln k)}$.

Let $f(x) = \frac{1}{\sqrt{x}\ln(\ln x)}$.

Thus, $f'(x) = \frac{(-2-\ln(x)\ln(\ln(x)))}{(2x(3/2)\ln(x)\ln^2(\ln(x)))} < 0, \forall x \in [3, \infty)$. Thus, b_k is monotone decreasing.

$\lim_{x \to \infty}^{\infty} \frac{1}{\sqrt{x}\ln(\ln x)} = 0$

Thus, by the alternating series test, we know that $\sum_{k=3}^{\infty} \frac{(-1)^k}{\sqrt{k}\ln(\ln k)}$ converges.

Next, we will test $|b_k| = \frac{1}{\sqrt{k}\ln(\ln k)}$ for convergence.

We will use the comparison test with $a_k = \frac{1}{n}$.

$\frac{1}{\sqrt{k}\ln(\ln k)} > \frac{1}{\sqrt{k}\ln k} > \frac{1}{k^{3/2}} > \frac{1}{k}, \forall k \in [3, \infty)$.

Therefore, by the comparison test, we know that $|b_k| = \frac{1}{\sqrt{k}\ln(\ln k)}$ diverges for all $k \in [3, \infty)$.

Therefore, $\sum_{k=3}^{\infty} \frac{(-1)^k}{\sqrt{k}\ln(\ln k)}$ converges conditionally.

7.3.6f

Test the following series for absolute and conditional convergence: $\sum\limits_{k=1}^{\infty} \frac{(-1)^{k+1}k^k}{(k+1)^{k+1}}$.

First, we will use the alternating series test:

Let $b_k = \frac{k^k}{(k+1)^{k+1}}$.

Let $f(x) = \frac{x^x}{(x+1)^{x+1}}$.

Then $f'(x) = \frac{(x^x)(\ln(x) - \ln(x+1))}{(x+1)^{x+1}} < 0, \forall x \in [1, \infty)$. Therefore, b_k is monotone decreasing.

$\lim\limits_{k\to\infty} \frac{k^k}{(k+1)^{k+1}} = \lim\limits_{k\to\infty} \frac{k^k}{(k+1)^k(k+1)} = 0$

Therefore, by the alternating series test, $\sum\limits_{k=1}^{\infty} \frac{(-1)^{k+1}k^k}{(k+1)^{k+1}}$ converges.

Next, we will evaluate $|\sum\limits_{k=1}^{\infty} \frac{(-1)^{k+1}k^k}{(k+1)^{k+1}}| = \sum\limits_{k=1}^{\infty} \frac{k^k}{(k+1)^{k+1}}$ for convergence.

We will use the comparison test with $a_k = \frac{1}{k}$.

$\lim\limits_{k\to\infty} \frac{\frac{k^k}{(k+1)^{k+1}}}{\frac{1}{k}} = \lim\limits_{k\to\infty} \frac{k^{k+1}}{(k+1)^{k+1}} = \frac{1}{e}$

$0 < \frac{1}{e} < \infty$, therefore by the limit comparison test, $\sum\limits_{k=1}^{\infty} \frac{k^k}{(k+1)^{k+1}}$ diverges.

Therefore, $\sum\limits_{k=1}^{\infty} \frac{(-1)^{k+1}k^k}{(k+1)^{k+1}}$ converges conditionally.

7.3.6h

Test the following series for absolute and conditional convergence: $\sum\limits_{k=1}^{\infty} (-1)^{k+1} \sin(\frac{1}{k})$.

We will use the alternating series test:

Let $b_k = \sin(\frac{1}{k})$.

Since $0 \leq \frac{1}{k} \leq \frac{\pi}{2}$ for $k \in \mathbb{N}$ and sin is monotonically decreasing over $[\frac{\pi}{2}, 0]$, we know that b_k is monotonically decreasing.

$\lim_{k\to\infty} b_k = \lim_{k\to\infty} \sin(\frac{1}{k} = \sin(\lim_{k\to\infty} \frac{1}{k}) = \sin(0) = 0$

Therefore, by the alternating series test, $\sum_{k=1}^{\infty} (-1)^{k+1} \sin(\frac{1}{k}) < \infty$.

$|\sum_{k=1}^{\infty} (-1)^{k+1} \sin(\frac{1}{k})| = \sum_{k=1}^{\infty} \sin(\frac{1}{k})$

We will use the limit comparison test:

Let $a_k = \frac{1}{k}$ and $b_k = \sin(\frac{1}{k})$.

$\lim_{k\to\infty} \frac{b_k}{a_k} = \lim_{k\to\infty} \frac{\sin(\frac{1}{k}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{\sin(k)}{k} = 1$

Since, $0 < 1 < \infty$, we know that b_k converges if and only if a_k coverges.

a_k is the harmonic series, which is divergent. Thus, $b_k = \sum_{k=1}^{\infty} \sin(\frac{1}{k})$ diverges.

Therefore $\sum_{k=1}^{\infty} (-1)^{k+1} \sin(\frac{1}{k})$ converges conditionally.

7.4.1b

Determine if the following sequence is in l^2*:* $\{\frac{1}{\sqrt{k}\ln k}\}_{k=2}^{\infty}$.

Let $a_k = \frac{1}{\sqrt{k}\ln k}$.

$||a_k||_2^2 = \sum_{k=2}^{\infty} a_k^2 = \sum_{k=2}^{\infty} (\frac{1}{\sqrt{k}\ln k})^2 = \sum_{k=2}^{\infty} \frac{1}{k{*}\ln^2(k)}$

We will use the integral test to test for convergence:

Let $f(x) = \frac{1}{x\ln^2(x)}$.

$f'(x) = -\frac{\ln(x)+2}{x^2{*}\ln^3(x)} < 0$ for all $x \in [2, \infty)$.

Therefore, $f(x)$ is monotone decreasing.

$\int_2^{\infty} f(x)dx = \int_2^{\infty} \frac{1}{x\ln^2(x)} = \lim_{n\to\infty} -\frac{1}{\ln x} = 0 < \infty$

Therefore, $\sum_{k=2}^{\infty} (\frac{1}{\sqrt{k}\ln k})^2$ converges and the sequence $\{\frac{1}{\sqrt{k}\ln k}\}_{k=2}^{\infty} \in l^2$.

7.4.1c

Determine if the following sequence is in l^2: $\{\frac{\ln k}{\sqrt{k}}\}_{k=2}^{\infty}$.

Let $a_k = \frac{\ln k}{\sqrt{k}}$.

$||a_k||_2^2 = \sum_{k=2}^{\infty} a_k^2 = \sum_{k=2}^{\infty} (\frac{\ln(k)}{\sqrt{k}})^2 = \sum_{k=2}^{\infty} \frac{\ln^2(k)}{k}$

We will use the comparison test to test for convergence:

Let $b_k = \frac{1}{k}$, which is the harmonic series and, thus, divergent.

We know that $0 \leq \frac{1}{k} \leq \frac{ln^2(k)}{k}$ for $k > e$ since $\ln^2(k) > 1$ when $k > e$.

Therefore, by the comparison test, since $\sum_{k=2}^{\infty} \frac{1}{k}$ diverges, $\sum_{k=2}^{\infty} \ln^2(k)k$ diverges.

Thus, the sequence $\{\frac{\ln k}{\sqrt{k}}\}_{k=2}^{\infty}$ is not in l^2.

7.4.2b

Determine all values of $p \in \mathbb{R}$ such that the given sequence is in l^2: $\{\frac{k^p}{p^k}\}_{k=1}^{\infty}$.

Let $a_k = \frac{k^p}{p^k}$.

$||a_k||_2^2 = \sum_{k=1}^{\infty} (\frac{k^p}{p^k})^2 = \sum_{k=1}^{\infty} \frac{k^{2p}}{p^{2k}}$.

We will first use the root test:

$\limsup_{k\to\infty} \sqrt[k]{\frac{k^{2p}}{p^{2k}}} = \limsup_{k\to\infty} \frac{\sqrt[k]{k^{2p}}}{p^2} = \frac{1}{p^2}(\limsup_{k\to\infty}(k^{1/k}))^{2p} = \frac{1}{p^2} * 1 = \frac{1}{p^2}$

Therefore, by the root test, we know that $\sum_{k=1}^{\infty} (\frac{k^p}{p^k})^2$ diverges if $p < 1$ and converges if $p > 1$

Now, we must examine the case when $p = 1$.

If $p = 1$, $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k^2}{1^{2k}} = \sum_{k=1}^{\infty} k^2$, which we know diverges.

Therefore, $\sum_{k=1}^{\infty} (\frac{k^p}{p^k})^2$ converges if and only if $p > 1$.

Thus the sequence $\{\frac{k^p}{p^k}\}_{k=1}^{\infty}$ is in l^2 if and only if $p > 1$.

7.4.3

If $\{a_k\} \in l^2$, prove that $\lim_{k\to\infty} a_k = 0$.

$\{a_k\} \in l^2$ implies that $||a_k||_2 = \sqrt{\sum |a_k|^2} < \infty$.

If $\sqrt{\sum |a_k|^2} < \infty$, $\sum |a_k|^2 < \infty$

$\sum |a_k|^2 < \infty$ implies $\lim_{k\to\infty} |a_k|^2 = 0$, by the definition of convergence.

Which implies that $\lim_{k\to\infty} |a_k| = 0$ and $\lim_{k\to\infty} a_k = 0$

7.4.6

a.)*Suppose that $\{p_k\}_{k=1}^{\infty}$ is a sequence in $\mathbb{R}^n$, where for each $k \in \mathbb{N}, p_k = (p_{k,1}, \cdots, p_{k,n})$. Prove that the sequence $\{p_k\}$ converges to $p = (p_1, \cdots, p_n)$ in the norm $|| \ ||_2$ if and only if $\lim_{k\to\infty} p_{k,i} = p_i$ for all $i = 1, \cdots, n$.*

Assume that $\lim_{k\to\infty} p_k = p$ Given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, so that $||p_n|| = \sqrt{(p_1 - p_{1,k})^2 + \cdots + (p_n, k)^2} < \epsilon$

For a fixed i,
$|p_i - p_{i,k}| = \sqrt{(a_i - a_{i,n})^2} \le ||p_n|| = \sqrt{(p_1 - p_{1,k})^2 + \cdots + (p_n, k)^2} < \epsilon$

Thus, for each $\epsilon > 0$, $\lim_{n\to\infty} p_{i,k} = p_i$.

Now, for the other direction, let $\lim_{n\to\infty} p_i, k = p_i$.

Given $\epsilon > 0$, for each i, let $\epsilon_i = \frac{\epsilon}{\sqrt{n}}$ and choose $N_i \in \mathbb{N}$ such that $|p_i - p_{i,k}| < \epsilon_1$.

Let $N = \max\{N_1, \cdots, N_n\}$.

Then, for every $i \in [0, n]$, we have $(p_i - p_{i,k})^2 < \epsilon_1^2$ when $k > N$.

Therefore $||p_k|| = \sqrt{(a_1 - a_{1,k})^2 + \cdots + (a_n - a_{n,k})^2} < \sqrt{\epsilon_1^2 + \cdots + \epsilon_1^2} < \sqrt{n}\epsilon_1 = \epsilon$.

Thus the sequence $\{p_k\}$ converges to $p = (p_1, \cdots, p_n)$ in the norm $|| \ ||_2$ if and only if $\lim_{k\to\infty} p_{k,i} = p_i$ for all $i = 1, \cdots, n$.

b.) *Prove that every bounded sequence in* $\mathbb{R}^n$ *has a convergent subsequence.*

Let $\{a_k\}$ be a bounded sequence in $\mathbb{R}^n$.

Since $\{x_k\}$ is bounded, there exists $a_1, b_1 \in \mathbb{R}^n$ such that $x_k \in [a_1, b_1], \forall k \in \mathbb{N}$.

Let $I_k = [a_k, b_k]$ such that $I_{k+1} \subseteq I_k$ and $(b_{k+1} - a_{k+1} = \frac{b_k - a_k}{2} = \cdots = \frac{b_1 - a_1}{2^k} \to 0$.

Then, by the Nested Interval Theorem, there exists some x_0 such that x_0 is in the intersection of all $I_k, k \in \mathbb{N}$.

We will construct a subsequence by choosing $x_{l_{k-1}} \in [a_{k-1}, b_{k-1}], x_{l_k} \in [a_k, b_k]$ such that $l_k > l_{k-1}$ for all $k \in \mathbb{N}$.

By design, we have $a_k \leq x_{l_k} \leq b_k$ with $a_k \to 0$ and $b_k \to 0$. Thus, by the squeeze theorem, $x_{l_k} \to 0$.

Therefore, every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

7.4.7

For each $n \in \mathbb{N}$*, let* e_n *be the sequence in* l^2 *defined by*

$$e_n(k) = \begin{cases} 0, & \textit{if } k \neq n \\ 1, & \textit{if } k = n. \end{cases}$$

Show that $||e_n - e_m||_2 = \sqrt{2}$ *if* $n \neq m$*. (Remark: The sequence* $\{e_n\}$ *is a bounded sequence* l^2 *with no convergent subsequence. Thus the Bolzano-Weierstrass theorem (2.4.11) fails in* l^2*.*

We know that for $n \in \mathbb{N}$, there is only one instance in which $n = m$, so $\sum e_n = 1$ and $\sum e_m = 1$.

In addition, for the $\sum e_n e_m$, since n and m are different, we know the sum will always be 0.

Therefore,
$||e_n - e_m||_2 = \sqrt{\sum (e_n - e_m)^2} = \sqrt{\sum e_n^2 - 2e_n e_m + e_m^2} = \sqrt{1+1} = \sqrt{2}$