Modern Algebra
Homework 7a
Chapter 7
Read 7.1-7.3
Complete 7.7, 7.8, 7.9, 7.12, 7.13, 7.17, 7.18c-h, 7.24
Proof

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7.7

a.) *Use direct product notation to describe the group depicted by the following Cayley diagram*

The group depicted is $C_2 \times C_2 \times C_3$.

b.) *The group C_10 is a direct product. What are its factors?*

The factors of C_{10} are C_2 and C_5.

c.) *Is the group depicted by the following Cayley diagram a direct product group? Justify your answer.*

No, the group depicted is a not direct product group.

7.8

a.) *If A and B are Abelian is $A \times B$ Abelian?*

Yes, the direct product of abelian groups is also abelian.

b.) *Justify your answer to (a) visually. If you answered yes, give evidence by explaining why the direct product process for two abelian Cayley diagrams* *must produce an abelian Cayley diagram. If you answered no, give Cayley diagrams for abelian groups A and B and the corresponding non-abelian group $A \times B$.*

The Cayley diagram for a group that is a direct product of abelian groups must also be abelian because the abelian factor groups will form subgroups in which the actions commute and the actions connecting the subgroups will also commute. Thus, every action will commute.

c.)*Justify your answer to (a) algebraically (either by reference to the groups' binary operations or their multiplication tables).*

Let G and H be two abelian groups. The direct product $G \times H$ is the set of all ordered pairs $\{(g,h)|g \in G, h \in H\}$ with the operation $(g_1,h_1)*(g_2,h_2) = (g_1g_2, h_1, h_2)$.

Let $(g,h),(g',h') \in G \times H$ where $g, g' \in G$ and $h, h' \in H$. Then $(g,h)(g',h') = (gg', hh') = (g'g, h'h) = (g',h')(g,h)$.

Therefore, if G and H are abelian, $G \times H$ must also be abelian.

d.)*If A is non-abelian, what can you conclude about $A \times B$?*

If A is non-abelian, then we know that the direct product $A \times B$ must also be non-abelian.

e.)*Justify your answer to (d) visually (by reference to Cayley diagrams).*

In the Cayley diagram for the direct product group, the actions in the subgroup A will not commute. This can be seen visually by attempting actions in different orders. Thus, not all actions will commute.

f.)*Justify your answer to (d) algebraically (either by reference to the groups' binary operations of their multiplication tables).*

Let G and H be two groups and G be non-abelian. The direct product $G \times H$ is the set of all ordered pairs $\{(g,h)|g \in G, h \in H\}$ with the operation $(g_1,h_1)*(g_2,h_2) = (g_1g_2, h_1, h_2)$.

Let $(g,h),(g',h') \in G \times H$ where $g, g' \in G$ and $h, h' \in H$. Then $(g,h)(g',h') = (gg', hh')$. Since G is non abelian, the binary operation in the first part of the resultant coordinate cannot commute. Therefore, the direct product cannot commute and cannot be abelian.

7.9

a.)*Reorganize the Cayley diagram for Q_4 to show the subgroup $\langle i \rangle$ and its left cosets. Is $\langle i \rangle$ a normal subgroup of Q_4? $\langle i \rangle = \{-i, 1, i, -1\}$.*

Yes, it is a normal subgroup.

b.)*Let's determine whether Q_4 is a direct product of $\langle i \rangle$ with some other subgroup $A < Q_4$. What size must A be?*

If Q_4 is a direct product with $\langle i \rangle$ with one of its factors, the other factor must be order 2 since $\langle i \rangle$ is order 4 and the order of the direct product is the order of the factors multiplied together.

c.)*Based on part (b), what are the possibilities for A?*

Based in the order requirment, the only possibility for A is $\langle -1 \rangle$, which is the only subgroup of Q_4 with order 2.

d.)*Is Q_4 a direct product $\langle i \rangle \times A$ for some A? If so, what is A? If not, why not?*

No, there is no group A such that $Q_4 = \langle i \rangle \times A$. Since $|Q_4| = 8$ and $|\langle i \rangle| = 4$, $|A| = 2$. All groups of order 2 are abelian and $\langle i \rangle$ is abelian. Therefore, the direct product of $\langle i \rangle \times A$ would be abelian, since the direct product of two abelian groups must be abelian. Since Q_4 is non-abelian, it cannot be a direct product group.

7.12

Prove that $A \triangleleft A \times B$ and $B \triangleleft A \times B$. You may find the equations on page 128 useful for an algebraic argument or Figure 7.14 for a visual one.

We know that since $A, B < A \times B$ by construction since
$A \times B = \{(a, b) | a \in A, b \in B\}$.

$A \triangleleft A \times B$ if and only if $gA = Ag$ for all $g \in G = A \times B$.

7.13

Draw a representative portion of the infinite Cayley diagram for the group $\mathbb{Z}^2$.

7.17

Consider the quotient taken in Figure 7.23.

a.) *What is the subgroup by which the quotient is taken? Where in the figure can you see that subgroup?*

The subgroup is V_4.

b.) *What is the order of that subgroup? How does the figure show that order?*

The order of the subgroup V_4 is 4. It is demonstrated in the figure by the four nodes in each of the large, grey clusters.

c.) *What is the index of that same subgroup? How does the figure show that index?*

The index of the subgroup is 3. This is demonstrated in the figure by the three large, grey clusters.

d.) *Does* A_4 *have any subgroups of order 3? How does the figure show you such a subgroup, or show you that there are not any?*

A_4 has four subgroups of order 3. The figure shoes these subgroups by the circles generated by the blue arrows.

e.) *Can A_4 be divided by any of its other subgroups?*

A quotient can only be taken if the subgroup is normal. The only subgroups of A_4 that are normal are the subgroups of order 4 which are all isomorphic to V_4. Therefore, A_4 can be divided only by its subgroups of order 4.

7.18

For each of the following H and G with $H < G$, attempt the quotient process from definition 7.5. If it succeeds, show a diagram like figure 7.20 and state the name of the quotient group. If the quotient operation reveals a direct or semidirect product structure, say which it is and name the factors. If the quotient operation fails, show a diagram like Figure 7.26.

c.) $G = C_{10}, H = \langle 2 \rangle$.

$G/H = C_{10}/\langle 2 \rangle \simeq C_{10}/C_5 \simeq C_2$.

d.) $G = D_4, H = \langle r^2 \rangle$.

$G/H = D_4/\langle r^2 \rangle \simeq V_4$

e.) $G = D_4, H = \langle f \rangle$.

$G/H = D_4/\langle f \rangle \simeq C_4$.

f.) *The group G shown in the Cayley diagram below, with H standing for the two-element subgroup generated by the green arrow.*

The subgroup is not normal and the resulting Cayley diagram from the quotient attempt is not regular. Therefore, the quotient cannot be taken.

g.) *The group G shown in the same Cayley diagram (above), but this time with H standing for the two-element subgroup generated by the blue arrow.*

The subgroup is not normal and the resulting Cayley diagram from the quotient attempt is not regular. Therefore, the quotient cannot be taken.

h.) *The group G shown in the Cayley diagram below (sometimes called $G_{4,4}$) with H standing for the two element subgroup generated by the red arrow.*

$G/H \simeq C_4 \times C_2$

7.24

Recall the group $\mathbb{Q}$ (under addition) and the group $\mathbb{Q}*$ (under multiplication) introduced in exercise 4.33.

a.)*Describe the quotient group $\frac{\mathbb{Q}}{\langle 1 \rangle}$.*

Since $\langle 1 \rangle = \{n(1) : n \in \mathbb{Z}\}$, we can consider the quotient group the group of rationals mod 1.

b.)*Describe the quotient group $\frac{\mathbb{Q}*}{\langle -1 \rangle}$.*

Since $\langle 1 \rangle = \{n(-1) = n : -n \in \mathbb{Z}\}$, the quotient group is a group of positive or negative rationals.

Proof

Recall that G/H is the set of left cosets of H in G. We defined a binary operation G/H of left cosets by $aH * bH = abH$. In this exercise, you will see further motivation for this definition. Given that $a, b \in G$, define the sets $aHbH = \{ah_ibh_2 : h_1, h_2 \in H\}$ and $abH = \{abh : h \in H\}$. Prove that if $H \triangleleft G$, then $aHbH = abH$ (show that an arbtrary element pf abH s in $aHbH$ and vice-versa). Comment on how this relates to quotient groups.