Modern Algebra
Homework 7b
Chapter 7
Read 7.1-7.3
Complete 7.7, 7.8, 7.9, 7.12, 7.13, 7.17, 7.18c-h, 7.24
Proof

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Proof 1

Prove or disprove the following statements (without refering to Cayley diagrams). To disprove something, all you need to do is find a single example where it fails.

a.)*Every subgroup of an abelian group is normal.*

If $H \triangleleft G$ and $g \in G, h \in H$, then $ghg^{-1} = hgg^{-1} = he = h \in H$ and H is normal.

b.)*Every quotient of an abelian group is abelian.*

Let H be any subgroup of some abelian group G. Since G is abelian, $H \triangleleft G$.

Thus, G/H exists for all H.

Let $X = xH, Y = yH$ for $x, y \in G$, where X is the left coset and Y is the right coset of H.

$XY = xHyH = xyHH = xyH = yxH = yHxH = YX$.

Therefore, G/H is abelian.

c.)*If $K \triangleleft H \triangleleft G$, then $K \triangleleft G$.*

This is false. Take $G = D_8, K = \langle f \rangle$ and $H = \langle r^2, f \rangle$. Then $K \triangleleft H$ and $H \triangleleft G$, but $K \ntriangleleft G$ since $rfr^{-1} \neq e$ or$f$.

d.)*If $K \leq H \leq G$ and $K \triangleleft G$, then $K \triangleleft H$.*

Let $g_1 \in C_g(H)$. Then $g_1 h = hg_1 \forall h \in H$. Let $g \in G$.

Since $H \triangleleft H$, $g^{-1}hg \in H$ and $gh_1 = hg$ and $g^{-1}h_1 = hg^{-1}$.

Then, $(g^{-1}g_1g)h = g^{-1}g_1(gh) = g^{-1}g_1(h_1g) = g^{-1}(g_1h_1)g = g^{-1}(h_1g_1)g = (g^{-1}h)g_1g = (hg^{-1})g_1g = h(g^{-1}g_1g)$

Therefore $g^{-1}g_1g \in C_G(H)$ and $K \triangleleft H$.

Proof 2

Recall that the center of a group G is the set $Z(G) = \{z \in G | gz = zg, \forall g \in G\} = \{z \in G | gzg^{-1} = z, \forall g \in G\}$.

a.)*Prove that $Z(G)$ is a subgroup of G. (That is, show that it contains the identity, inverses, and is closed under the group operation.)*

For any $g \in G$, $eg = g = ge$ and $e \in Z(G)$ and $Z(G) \neq 0$, where e is the identity element of G.

Let $a, b \in Z(G)$, for any $g \in G$, then $(ab)g = a(bg) = a(gb) = (ag)b = (ga)b = g(ab)$ thus $ab \in Z(G)$ and $Z(G)$ is closed under multiplication.

Let $a \in Z(G)$, for any $g \in G$, then $a^{-1}g = (g^{-1}a)^{-1} = ga^{-1}$, and $a^{-1} \in Z(G)$.

Therefore, $Z(G)$ is a subgroup of G.

b.)*Prove that $Z(G)$ is a normal subgroup of G. (That is, show that for any $x \in Z(G)$, the element $gxg^{-1} \in Z(G)$.))*

We already know that $Z(G)$ is a subgroup of G.

Let $g \in G$ and $a \in Z(G)$, then $g^{-1}ag = ag^{-1}g = a \in Z(G)$. Thus, Z is a normal subgroup.

7.25c

Compute the normalizer of H in G for the following case. None of these requires drawing; all are possible by thinking (and perhaps using your mind's eye: G is any group and $H = \{e\}$.

$N_G(H) = \{x \in G | xHx^{-1} = H\} = G$, since for all $g \in G$, $gHg^{-1} = g\{e\}g^{-1} = \{geg^{-1}\} = \{gg^{-1}\} = \{e\} = H$.

7.25d

Compute the normalizer of H in G for the following case. None of these requires drawing; all are possible by thinking (and perhaps using your mind's eye: $G = D_n$, and $H = \langle r \rangle$.

D_n has the form $< f, r | f^2 = r^n = e, frf^{-1} = r^{-1} >$.

$N_{D_n} = \{x \in G | xHx^{-1} = H\}$, where $H = G$.

$N_{D_n} = \{x \in G | x\langle r \rangle x^{-1} = \langle r \rangle\} = \langle r \rangle$.

7.26c

Illustrate the normalizer of H in G in he following case, as Figure 7.29 *did for $\langle f \rangle < D_6$: $G = D_4, H = \langle f, r^2 \rangle$.*

7.26d

Illustrate the normalizer of H in G in he following case, as Figure 7.29 *did for $\langle f \rangle < D_6$: $G = D_5, H = \langle f \rangle$.*

7.27

From what you observed as you did exercise 7.26, can you say for which n the normalizer $N_{D_n}(\langle f \rangle) \neq \langle f \rangle$.

For $N_{D_n}(H) = H$, H must be a normal subgroup of D_n. There are no n for which D_n has a normal subgroup $\langle f \rangle$.

7.29

Show that all of an element's conjugates have the same order as the element.

Let $g \in G$ with order n such that $g^n = e$ and $g^k \neq e$ for $0 < k < n$where e is the identity element of G.

Let $a \in G$ be a conjugate of g such that $a = bgb^{-1}$ for some $b \in G$.

For $k \in (0, n)$, $a^k = (bgb^{-1})^k = b(g^k)b^{-1}$

If $a^k = e$, then $e = bg^kb^{-1}$ which implies $b^{-1}eb = g^k$ and $e = g^k$, which is a contradiction since our assumption states $0 < k < n, g^k \neq e$.

Since, $a^n = (bgb^{-1})^n = b(g^n)b^{-1} = beb^{-1} = bb^{-1} = e$.

Therefore if g has order n then a has order n.

7.32

Let c and t stand for the permutations shown below, members of S_n: $c =$ $1\ 2\ 3\ 4\ \cdots\ n$ and $t = 1\ 2\ 3\ \cdots\ n$. Thus c stands for a cycle of n numbers in order and t stands for the interchange of just the numbers 1 and 2, leaving the rest alone. This exercise determines what elements of the subgroup $\langle c, t \rangle$ of S_n contains.

a.) *What is the conjugate of t by c, written ctc^{-1}? What is the conjugate of t by c^k, for any $k < n$?*

The conjugate of t by c, where$t = (1\ 2)$, is $ctc^{-1} =$ $((1\ 2\ 3\ \cdots\ n)(1), (1\ 2\ 3\ \cdots\ n)(2))$.

The conjugate of t by c^k for any k up to n, $t = (1\ 2)$, therefore $c^ktc^{-k} = ((1\ 2\ 3\ \cdots\ n)(1), (n\ \cdots\ 3\ 2\ 1)(2))$.

b.) *All the conjugates from part (a) are in* $\langle c, t\rangle$. *Describe that set of conjugate elements.*

$Cl(t) = \{h \in G : h = ctc^{-1} forsomec \in G\} = \{(1\ 2\ 3\ \cdots\ n)\}$.

c.) *What is the conjugate of t b the following permutation, which interchanges just the numbers* 2 *and* 3*, leaving the rest alone?* 1 2 3 4 $\cdots$ n. *How could you use two of the elements in* $\langle c, t\rangle$ *to create a permutation that swaps any two numbers from* 1 *to* n*, leaving the rest alone?*

The conjugate of t by c, is
$t = (2\ 3)$, is $ctc^{-1} = ((1\ 2\ 3\ \cdots\ n)(2), (1\ 2\ 3\ \cdots\ n)(3))$.

d.) *Describe the set of elements that part (c) shows to be members of* $\langle c, t\rangle$.

$Cl(t) = \{h \in G : h = ctc^{-1}$ for some $c \in G\} = \{(1\ 2\ 3\ \cdots\ n)\}$.

e.) *What permutation is obtained by doing* t *followed by the permutation shown in part (c)? How could you create any cyclic permutation using just elements of* $\langle c, t\rangle$*?*

The permutation obrained is $(1\ 2)((1\ 2\ 3\ \cdots\ n)(2), (1\ 2\ 3\ \cdots\ n)(3))$.

f.) *All permutations can be broken into a sequence of non-overlapping cyclic permutations, as in the following example:* 1 2 3 4 5 = (1 2 3 4 5)$*$(1 2 3 4 5). *How does this determine the subgroup* $\langle c, t\rangle$ *of* S_n*? What is that subgroup?*

The subgroups $\langle c, t\rangle < S_n$ is $\langle (1\ 2\ 3\ \cdots\ n)(1\ 2), (1\ 2)(1\ 2\ 3\ \cdots\ n)\rangle$.

7.33a

Compute the class equation for the first few dihedral groups D_n *with* n *odd, until you notice a pattern. State the pattern and give some justification for it.*

$|D_3| = 1 + 2 + 3 = 6$

$|D_5| = 1 + 2 + 2 + 5 = 10$.

$|D_7| = 1 + 2 + 2 + 2 + 7 = 14$

Therefore, $|D_n| = 1 + (2 * \frac{n-1}{2}) + n = 2n$

7.33b

Compute the class for the first few dihedral groups with D_n with n even, until you notice a pattern. State the pattern and give some justification for it.

$|D_4| = 1 + 1 + 2 + 2 + 2 = 8$

$|D_6| = 1 + 1 + 2 + 2 + 3 + 3 = 12$

Therefore, $|D_n| = 1 + 1 + (2 * \frac{n-2}{2}) + \frac{n}{2} * 2$