LINEAR PROGRAMMING

Homework 7

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1. *Your friend is taking a Linear Programming course at another university and for homework she is asked to solve the following LP:*

   Primal:

$$\begin{aligned} \max\ & 8x_1 - 9x_2 + 12x_3 + 4x_4 + 11x_5 \\ \text{s.t.}\ & 2x_1 - 3x_2 + 4x_3 + x_4 + 3x_5 \leq 1 \\ & x_1 + 7x_2 + 3x_3 - 2x_4 + x_5 \leq 1 \\ & 5x_1 + 4x_2 - 6x_3 + 2x_4 + 3x_5 \leq 22 \\ & x_1.x_2.x_3.x_4, x_5 \geq 0 \end{aligned}$$

   She thinks that after doing tons of calculations and writing down Simplex tableaus with fractions as nasty as $\frac{34}{271}$ she finally found the optimal solution $x = (0, 2, 0, 7, 0)^T$. You, of course, have been paying attention in the recent lectures about duality and complementary slackness and are able to check very quickly whether this is in fact the case. And when you find out, break it to her gently...

   Dual:

$$\begin{aligned} \min\ & y_1 + y_2 + 22y_3 \\ \text{s.t.}\ & 2y_1 + y_2 + 5y_3 \geq 8 \\ & -3y_1 + 7y_2 + 4y_3 \geq -9 \\ & 4y_1 + 3y_2 - 6y_3 \geq 12 \\ & y_1 - 2y_2 + 2y_3 \geq 4 \\ & 3y_1 + y_2 + 3y_3 \geq 11 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

   Proposed Primal Solution $x* = (0, 2, 0, 7, 0)^T$.

   We know from the complementary slackness theorem that a vector x* is optimal for the primal linear program (P) if and only if x* is feasible for (P) and some feasible solution for the dual linear program (D) satisfies the following complimentary slackness conditions:

$$
\begin{aligned}
y_1 * (2x_1 - 3x_2 + 4x_3 + x_4 + 3x_5 - 1) &= 0 \\
y_2 * (x_1 + 7x_2 + 3x_3 - 2x_4 + x_5 - 1) &= 0 \\
y_3 * (5x_1 + 4x_2 - 6x_3 + 2x_4 + 3x_5 - 22) &= 0 \\
x_1 * (2y_1 + y_2 + 5y_3 - 8) &= 0 \\
x_2 * (-3y_1 + 7y_2 + 4y_3 + 9) &= 0 \\
x_3 * (4y_1 + 3y_2 - 6y_3 - 12) &= 0 \\
x_4 * (y_1 - 2y_2 + 2y_3 - 4) &= 0 \\
x_5 * (3y_1 + y_2 + 3y_3 - 11) &= 0
\end{aligned}
$$

We must first check to see whether or not $x*$ is feasible for the primal problem.

$$
\begin{aligned}
2(0) - 3(2) + 4(0) + (7) + 3(0) = -1 &\leq 1 \\
0 + 7(2) + 3(0) - 2(7) + 0 = 0 &\leq 1 \\
5(0) + 4(2) - 6(0) + 2(7) + 3(0) = 22 &\leq 22 \\
x* &\geq 0
\end{aligned}
$$

We see that $x*$ is feasible for the primal problem.

Since $x_2* = 2$, we know $-3y_1 + 7y_2 + 4y_3 + 9 = 0$.

Since $x_4* = 0$, we know $y_1 - 2y_2 + 2y_3 - 4 = 0$.

Since $x_1 * + 7x_2 * + 3x_3 * - 2x_{4*} + x_5 * - 1 = -1 \neq 0$, we know $y_2 = 0$.

This gives us the following system of equations:

$$
\begin{aligned}
-3y_1 + 7y_2 + 4y_3 + 9 &= 0 \\
y_1 - 2y_2 + 2y_3 - 4 &= 0 \\
y_2 &= 0
\end{aligned}
$$

Which has the unique solution $y* = (\frac{-221}{13}, 0, \frac{21}{2})^T$. However, $y_1* \leq 0$, which violates the nonnegativity constraint of the dual problem, meaning that $y*$ is infeasible. Therefore, we know that $x*$ is not an optimal solution.

Therefore, I would gently explain to my friend that there is an error in her calculations as her derived solution $x*$ cannot be optimal by the complementary slackness theorem since the corresponding dual solution $y*$ is infeasible for the dual problem. I would suggest that she attempt to solve the problem using either the revised simplex method or possibly AMPL.

2. *The labor union and management of a particular company have been negotiating a new labor contract. However, negotiations have now come to an impasse, with management making a final offer of a wage increase of* \$1.10 *per hour and the union making a final demand of*

$1.60 *per hour increase. Therefore, both sides have agreed to let an impartial arbitrator set the wage increase somewhere between the two figures. Each side must submit a sealed (i.e. secret) final proposal (rounded to a multiple of ten cents) for arbitration. Both sides know how the arbitrator will decide the outcome from arbitration for a given pair of proposals: the basic principle is that the side that moves the most from its previously final offer will either have its proposal accepted or the two sides settle on the average (whichever is better for the side that moves the most). More precisely, if the unions end up proposing an increase of no more than the proposal by management, the average of the two proposals is settled on. If both sides change their offers by the same ammount (say to* $1.50 *and* $1.20 *for labor and management respecitvely), then the average of the proposals is the final settlement. Otherwise, the side that moved the most has its proposal accepted. (For example, if the proposals are* $1.50 *and* $1.30 *for labor and management respectively, then the final settlement is* $1.30.*)*

(a) *Formulate this problem as a 2-player 0-sum game (i.e. a game where the amount one player pays is the amount the other player gets). Set up you matrix so that the union is the row player and the management is the column player.*

Union \ Management	1.10	1.20	1.30	1.40	1.50	1.60	**Worst(Min)**
1.60	1.35	1.20	1.30	1.40	1.50	1.60	1.20
1.50	1.50	1.35	1.30	1.45	1.50	1.55	1.30
1.40	1.40	1.40	**1.35**	1.40	1.45	1.50	**1.35**
1.30	1.30	1.30	1.30	1.35	1.40	1.45	1.30
1.20	1.20	1.20	1.20	1.30	1.35	1.40	1.20
1.10	1.15	1.20	1.25	1.30	1.35	1.35	1.10
Worst(Max)	1.50	1.40	**1.35**	1.45	1.50	1.55	

(b) *Find a saddle point in the matrix from (a).*

We see that if the union were to imply the maximin strategy that they would decide to propose $1.40, which would minimize their losses by maximizing the minimum pay increase to $1.35. Furthermore, if the management were to employ the similar minimax strategy, they would propose a wage increase of $1.30, which would minimize the maximum increase to $1.35.

Since the maximin and minimax are both wage increases of $1.35, we have a saddlepoint. This saddle point is an equilibrium point in that having reached this point, neither the management nor the union can make a unilateral move without harming its position.

3. *Sandy is a great fan of horse races. Having taken a probability course, she feels that betting a horse race can cause her great excitement. There are n horses participating in the next race where the rating bets are* $a_1, \ldots, a_n$ *respectively (i.e. if you bet*x_i *dollars on horse i and this horse wins the race, then you will get back* $x_i + a_i x_i$ *dollars so that your net payoff is* $a_i x_i$*). Though betting is exciting, Sandy also likes her money. Being risk averse in her nature, she wants to bet in a way that will maximize the return in the worst case scenario for her. Luckily, she has taken CSCI 628 as well so she knows that LP's can really help in deciding upon the optimal betting strategy. Write an LP that can find the optimal betting policy for*

Sandy, given that she is risk averse (you might find it useful to assume that Sandy has one unit of money to bet and that she needs to decide on how to divide this unit among bets on the n horses participating in the race).

In order to honor Sandy's risk averse nature, we want to make certain that we are maximizing the minimum return on her bets and thus will employ the maximin method.

$$
\begin{aligned}
&\max\ w \\
&\text{s.t.}\ \sum_{i=1}^{n} a_{ij}x_i \geq w, j = 1, 2, \ldots, n \\
&\qquad \sum_{i=1}^{n} x_i = 1 \\
&\qquad x_i \geq 0, i = 1, 2, \ldots, n
\end{aligned}
$$

Where $\sum\limits_{i=1} a_{ij}x_i$ is the payout for betting strategy j with horses i and w is the minimum return. The constraint $\sum\limits_{i=1}^{n} x_i = 1$ ensures that Sandy divides her entire unit of money among the horses for a specific betting strategy. This linear program will allow Sandy to optimize her betting strategy while minimizing her risk, thus allowing her to remain true to her risk averse nature.

4. *The game of Morra: Two players simultaneously throw out one or two fingers and call out their guess as to what the total sum of the outstretched fingers will be. If a player guesses right, but her opponent does not, then she receives a payment equal to her guess. In all other cases, it is a draw.*

   (a) *List all pure strategies for this game.*

   We know that a pure strategy is one complete response set that the player will employ throughout the game. By this definition, there are 8 pure strategies in the Two-Finger Morra, 4 for each player. Each player has the same 4 pure strategies available to them.

   We will let (i, j) represent the player's strategy with i representing the number of fingers the player chooses to display and j representing the player's guess for the total sum.

   Therefore the following strategies are available to each player:

$$
\begin{aligned}
&(1, 1) \\
&(1, 2) \\
&(2, 2) \\
&(2, 1)
\end{aligned}
$$

(b) *Write down the payoff matrix for this game.*

Row Player \ Column Player	(1,1)	(1,2)	(2,1)	(2,2)	**Worst(Min)**
(1,1)	0	2	-3	0	-3
(1,2)	-2	0	0	3	-2
(2,1)	3	0	0	-4	-4
(2,2)	0	-3	4	0	-3
Worst(Max)	3	2	4	3	

We see that the maximin $\neq$ minimax, therefore we know that there does not exist a pure-strategy saddle point.

(c) *Formulate the row player's problem as a linear programming problem.*

In order to formulate the row player's problem as a linear program, we must first make the payoff matrix non-negative. This is accomplished by adding the negative of maximum negative value in the payoff matrix, which, in this case, means that we will add 4 to each entry. This results in the following modified payoff matrix:

$$A = \begin{pmatrix} 4 & 6 & 1 & 4 \\ 2 & 4 & 4 & 7 \\ 7 & 4 & 4 & 0 \\ 4 & 1 & 8 & 4 \end{pmatrix}$$

It is important to note that while this modification increased the value of the payoff matrix by 4, it did not modify the relationships and thus will not alter the optimal strategy. Therefore, we can use this non-negative matrix A to formulate our LP for the row-player's optimal strategy.

Therefore, the linear program problem for the row-player's optimal strategy is as follows

$$\begin{aligned} \max\ & w \\ \text{s.t.}\ & \sum_{i=1}^{4} a_{ij} x_i \geq w, j = 1, \ldots, 4 \\ & \sum_{i=1}^{4} x_i = 1 \\ & x_i \geq 0, i = 1, \ldots, 4 \end{aligned}$$

(d) *What is the value of this game?*

We know that the value of the game is zero since the game is symmetric (i.e. the payoff matrix $A = -A^T$).

(e) *Find an optimal randomized strategy for the row player. You may use the AMPL file posted.*

AMPL Model File:

```
set ROWS;
set COLS;

param P {ROWS,COLS} default 0;

var x {ROWS} >= 0;
var w;

maximize worstcase_outcome: w;

subject to ineqs {j in COLS}:
sum{i in ROWS}P[i,j] * x[i] >= w;

subject to prob:
sum{i in ROWS} x[i] = 1;

data;
set ROWS := OneOne OneTwo TwoTwo TwoOne;
set COLS := OneOne OneTwo TwoTwo TwoOne;
param P:
OneOne OneTwo TwoTwo TwoOne :=
OneOne 4 6 1 4
OneTwo 2 4 4 7
TwoTwo 7 4 4 0
TwoOne  4 1 8 4;
```

AMPL Output:

```
Gurobi 5.6.3: optimal solution; objective 4
2 simplex iterations
ampl: display x;
x [*] :=
OneOne  0
OneTwo  0.571429
TwoOne  0
TwoTwo  0.428571
;
```

Therefore, an optimal randomized strategy for the row player is $x* = (0, 0.571429, 0, 0.428571)$.

(f) *Charecterize all optimal strategies for the row player and the column player and prove that the strategies that you list are indeed optimal.*

Since the game is symmetric (as described in part (d)), all optimal strategies for the row player and the column player are the same. That is all strategies are $x* = (0, 0.571429, 0, 0.428571)$. This is also supported by the fact that this is a zero value game.