Advanced Calculus II
Unit 8.1: 8.1.1a, 8.1.1b, 8.1.3, 8.1.4
Unit8.2: 8.2.2c, 8.2.7, 8.2.8a, 8.2.8c, 8.2.9a, 8.2.9e

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8.1.1a

Find the pointwise limits of the following sequences of functions on the given set: $\{\frac{nx}{1+nx}\}, x \in [0, \infty)$

$$\lim_{n\to\infty} \frac{nx}{1+nx} = \lim_{n\to\infty} \frac{x}{\frac{1}{n}+x} = \begin{cases} 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

8.1.1b

Find the pointwise limits of the following sequences of functions on the given set: $\{\frac{\sin(nx)}{1+nx}\}, x \in [0, \infty)$

$$\lim_{n\to\infty} \frac{\sin(nx)}{1+nx} = \lim_{n\to\infty} \frac{\frac{\sin(nx)}{nx}}{\frac{1}{nx}+1} = 0, \text{ since } \lim_{n\to\infty} \frac{\sin(nx)}{nx} = 0.$$

8.1.3

Consider the sequence $\{f_n\}$ *with* $n \geq 2$*, defined on* $[0, 1]$ *by*

$$f_n(x) = \begin{cases} n^2 x, & 0 \leq x \leq \frac{1}{n} \\ 2n - n^2 x, & \frac{1}{n} < x \leq \frac{2}{n} \\ 0, & \frac{2}{n} < x \leq 1 \end{cases}$$

a.) *Sketch the graph of* f_n *for* $n = 2, 3,$ *and* 4.

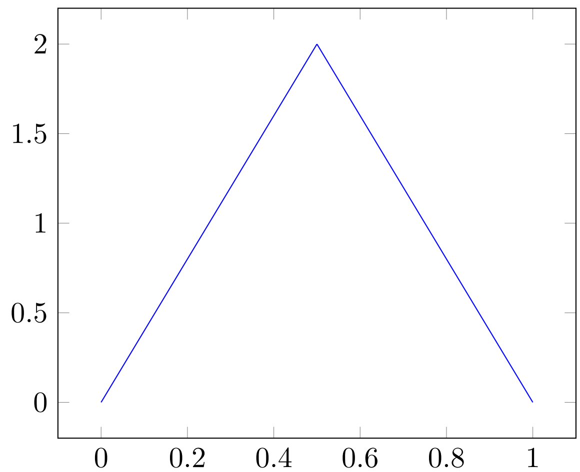

Figure 1: $n = 2$

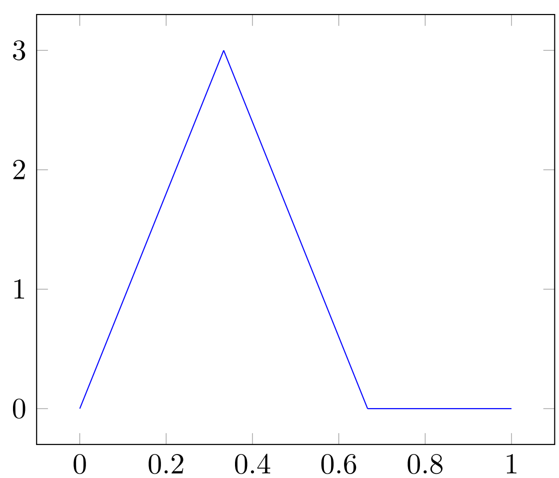

Figure 2: $n = 3$

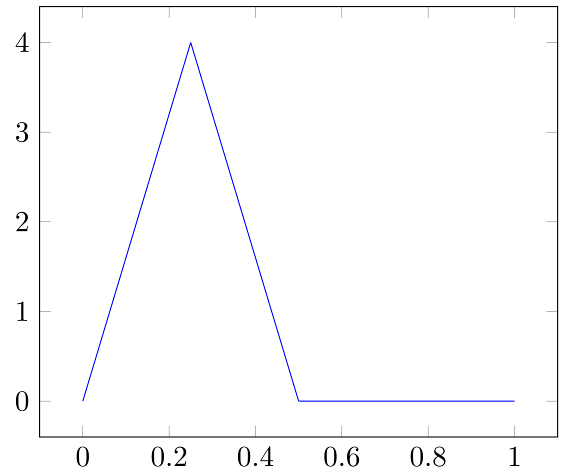

Figure 3: $n = 4$

b.)*Prove that* $\lim_{n\to\infty} f_n(x) = 0$ *for each* $x \in [0,1]$.

There are two cases: $x \in (0,1]$ and $x = 0$.

If $x = 0$, then $\lim_{n\to\infty} f_n(0) = 0$ for all n.

If $x \in (0,1]$, let $x > 2/N$ for some $N \in \mathbb{N}$, then $\lim_{n\to\infty} f_n(x) = 0$ for all $n > N$.

Therefore, $\lim_{n\to\infty} f_n(x) = 0$ for each $x \in [0,1]$.

c.)*Show that* $\int_0^1 f_n(x)dx = 1$ *for all* $n = 2, 3, \cdots$.

$$\int_0^1 f_n = \int_0^{\frac{1}{n}} n^2 x dx + \int_{\frac{1}{n}}^{\frac{2}{n}} (2n - n^2 x) dx + \int_{\frac{2}{n}}^1 0 dx = \int_0^{\frac{1}{n}} n^2 x dx + \int_{\frac{1}{n}}^{\frac{2}{n}} (2n - n^2 x) dx$$

$$= \frac{n^2 x^2}{2}\Big|_0^{\frac{1}{n}} + (2nx - \frac{n^2 x^2}{2})\Big|_{\frac{1}{n}}^{\frac{2}{n}}$$

$$= \frac{n^2(\frac{1}{n})^2}{2} + (2n\frac{2}{n} - \frac{n^2 x^2}{2}\frac{2}{n}) - ((2n\frac{1}{n} - \frac{n^2 x^2}{2}\frac{1}{n}) = \frac{1}{2} + \frac{1}{2} = 1$$

8.1.4

Let $g_n(x) = \frac{e^{-nx}}{n}, x \in [0, \infty), n \in \mathbb{N}$. *Find* $\lim_{n\to\infty} g_n(x)$ *and* $\lim_{n\to\infty} g_n'(x)$.

$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \frac{e^{-nx}}{n} = (\lim_{n\to\infty} e^{-nx})(\lim_{n\to\infty} \frac{1}{n}) = 0 * 0 = 0$

$g_n'(x) = -e^{-nx}$

$\lim_{n\to\infty} g_n'(x) = \lim_{n\to\infty} -e^{-nx} = 0.$

8.2.2c

Find examples of sequences $\{f_n\}$ *and* $\{g_n\}$ *that converge uniformly on a set* E*, but for which* $\{f_n g_n\}$ *does not converge uniformly on* E.

Let $\{f_n\} = \{\frac{1}{x}\}$ and $\{g_n\} = \{\frac{1}{n}\}$ on $(0, 1)$.

$M_{n_1} = \sup_{x\in 0,1} |f_n(x) - f(x)| = \sup_{x\in(0,1)} |\frac{1}{x} - \frac{1}{x}| = 0.$

Then $\lim_{n\to\infty} M_{n_1} = 0$ and f_n converges uniformly.

$M_{n_2} = \sup_{x\in 0,1} |g_n(x) - g(x)| = \sup_{x\in(0,1)} |\frac{1}{n} - 0| = \frac{1}{n}.$

Then $\lim_{n\to\infty} M_{n_2} = \lim_{n\to\infty} \frac{1}{n} = 0$ and g_n converges uniformly.

But, $\{f_n g_n\} = \{\frac{1}{xn}\}$ and $M_n = \sup_{x\in 0,1} |f_n(x)g_n(x) - f(x)g(x)| =$
$\sup_{x\in 0,1} |\frac{1}{xn} - 0| = \infty$

Therefore, $\lim_{n\to\infty} M_n = \infty$ and $\{f_n g_n\}$ does not converge uniformly.

8.2.8a

Show that the following series converges uniformly on the indicated interval: $\sum\limits_{k=1}^{\infty} \frac{1}{k^2+x^2}, 0 \leq x < \infty.$

$|\frac{1}{k^2+x^2}| \leq \frac{1}{k^2}$ for all $x \in (0, \infty)$ and $k \in \mathbb{N}$

$\sum\limits_{k=1}^{\infty} \frac{1}{k^2} < \infty$ since $\frac{1}{k^2}$ is a convergent p-series.

Therefore, by the Weierstrass M-Test, $\sum\limits_{k=1}^{\infty} \frac{1}{k^2+x^2}$ is uniformly convergent on $[0, \infty)$ since $\sum\limits_{k=1}^{\infty} \frac{1}{k^2} < \infty$.

8.2.8c

Show that the following series converges uniformly on the indicated interval: $\sum\limits_{k=1}^{\infty} k^2 e^{-kx}, 1 \leq x < \infty.$

$k^2 e^{-kx} \leq k^2 (\frac{1}{e})^k$ for all $x \geq 1$.

$\sum\limits_{k=1}^{\infty} k^2 (\frac{1}{e})^k < \infty$ since $\frac{1}{e} < 1$.

Therefore, by the Weierstrass M-Test, $\sum\limits_{k=1}^{\infty} k^2 e^{-kx}$ is uniformly convergent on $[1, \infty)$ since $\sum\limits_{k=1}^{\infty} k^2 (\frac{1}{e})^k < \infty$.

8.2.9a

Test the following series for uniform convergence on the indicated interval: $\sum_{k=1}^{\infty} \frac{\sin(2kx)}{(2k+1)^{3/2}}, x \in \mathbb{R}$.

$|\frac{\sin(2kx)}{(2k+1)^{3/2}}| < \frac{1}{(2k+1)^{\frac{3}{2}}} \leq \frac{1}{k^{\frac{3}{2}}}$ for all $x \in \mathbb{R}$.

$\sum_{k=1}^{\infty} \frac{1}{k^{\frac{3}{2}}} < \infty$ since $\frac{1}{k^{\frac{3}{2}}}$ is a convergent p-series.

Therefore, by the Weierstrass M-Test, $\sum_{k=1}^{\infty} \frac{\sin(2kx)}{(2k+1)^{3/2}} < \infty$ for $x \in \mathbb{R}$, since $\sum_{k=1}^{\infty} \frac{1}{k^{\frac{3}{2}}} < \infty$.

8.2.9e

Test the following series for uniform convergence on the indicated interval: $\sum_{k=1}^{\infty} \sin(\frac{x}{k^p}), p > 1, |x| \leq 2$.

$$\sum_{k=1}^{\infty} \sin(\frac{x}{k^p}) = \sum_{k=1}^{\infty} \sqrt{\frac{1-cos(x)}{k^p}} = \sum_{k=1}^{\infty} \frac{\sqrt{1-cos(x)}}{k^{\frac{p}{2}}}$$

$$M_n = \sup_{x \in (0,1)} |f_n(x) - f(x)| = \sup_{x \in (0,1)} |\frac{\sqrt{1-cos(x)}}{n^{\frac{p}{2}}} - 0| = \infty$$

Therefore, $\sum_{k=1}^{\infty} \sin(\frac{x}{k^p}), p > 1, |x| \leq 2$ does not converge uniformly.