Modern Algebra
Homework 8a
Chapter 8
Read 8.1-8.2
Complete 8.2, 8.3, 8.4, 8.5, 8.10, 8.12, 8.15, 8.16, 8.17, 8.36

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8.2

For each statement below, determine whether it is true or false

a.)*For any groups H and G, there is some homomorphism from H to G.*

True. There is always the trivial homomorphism which maps all elements of H to the identity element in G

b.)*For any groups H and G, there is some embedding of H to G.*

False. An embedding maps the domain to a copy of the domain in the codomain, which is not always possible. For example, $\phi : S_3 \to C_3$ maps only to the identy element in C_3 and is thus not an embedding.

c.)*Every homomorphism is either an embedding or a quotient map.*

True. If a mapping satisfies the conditions of a homomorphism it must either be an embedding or a quotient map.

d.)*Embeddings are those homomorphisms whose kernel is empty.*

False. For every homomorphism $\phi : G \to H$, the identity element in G must map to the identity element in H and is in the kernel.

e.) *When $A \cong B$, there is some isomorphism $i : A \to B$, and therefore there is also an isomorphism $j : B \to A$.*

True, because homomorphisms are reversible, the isomorphism that maps $A \to B$ reverted can map $B \to A$.

8.3

If $\phi : G \to H$ maps every element of G to the identity element of H, is ϕ a homomorphism?

Yes. It is the trivial homomorphism.

8.4

For each part below, list all homomorphisms (both embeddings and quotient groups) with the given domain and codomain. Does each collection of homomorphisms form a group, as collections of automorphisms do?

a.) *Domain C_3 and codomain C_2.*

$\phi(n) = e$

b.) *Domain C_2 and codomain C_3.*

$\phi(n) = e$

c.) *Domain and codomain both C_4.*

$\phi(n) = e$

$\phi(n) = n$

$\phi(n) = 2n$

$\phi(n) = 3n$

d.) *Domain C_2 and codomain V_4.*

$\phi(n) = e$

$\phi(n) = v^n$

$\phi(n) = h^n$

$\phi(n) = (hv)^n$

e.) *Domain and codomain both V_4.*

$\phi(v) = e, \phi(h) = e$

$\phi(v) = v, \phi(h) = h$

$\phi(v) = h, \phi(h) = v$

$\phi(v) = e, \phi(h) = h$

$\phi(v) = hv, \phi(h) = v$

8.5

Consider the function $\phi : \mathbb{Z} \to \mathbb{Z}$ by $\phi(n) = 2n$. Justify your answer to each of the following questions regarding ϕ.

a.) *Is it a homomorphism? If so, is it an embedding or a quotient map?*

It is a homomorphism because the identity in $\mathbb{Z}$ is mapped to the identity in $\mathbb{Z}$ and for all $a, b \in \mathbb{Z}$, $\phi(ab) = \phi(a)\phi(b)$. It is an embedding because $\mathbb{Z}$ is an infinite cyclic group, therefore the structure of $\mathbb{Z}$ can be replicated.

b.) *Would ϕ be a homomorphism if it were to use a different coefficient than 2? If so, what numbers could be used in place of 2?*

Any $n \in \mathbb{Z}$ could be used in place of 2 because $\mathbb{Z}$ is an infinite cyclic group.

c.) *What are $Ker(\phi)$ and $Im(\phi)$?*

The $Ker(\phi) = 0$ and the $Im(\phi) = \{n \in \mathbb{Z} : n \text{ is even }\}$.

8.10

For each part below, describe all the embeddings with the given domain and codomain. Chose one from each part (if available) to diagram.

a.)*Domain* C_2 *and codomain* V_4.

$\phi(n) = v^n$

$\phi(n) = h^n$

$\phi(n) = (hv)^n$

b.)*Domain* C_2 *and codomain* C_3.

There is no embedding of C_2 in C_3 because C_3 doesn't have a subgroup of size 2

c.)*Domain* C_2 *and codomain* C_4.

$\phi(n) = a^n$

$\phi(n) = a^{2n}$

d.)*Domain C_3 and codomain S_3.*

$\phi(n) = r^n$

e.)*Domain C_n and codomain $\mathbb{Z}$.*

$\phi(n) = 0$

$\phi(n) = rn, \forall r \in \mathbb{Z}$

f.)*Domain and codomain both $\mathbb{Z}$.*

$\phi(n) = 0$

$\phi(n) = n$

8.12

For parts (a) through (c), a group G is given together with a normal subgroup H. Illustrate not only the quotient map $q : G \to \frac{G}{H}$, but also illustrate the embedding $\phi : H \to G$ chained together so that $Im(\phi) = Ker(q)$.

a.)$H = C_3, G = C_6$.

b.)$H = C_3, G = S_3$.

c.)$H = V_4, G = A_4$.

Now answer each of the following questions about each of your answers to part (a) through (c).

d.) *What map θ into H would satisfy the equation $Im(\theta) = Ker(\theta)$? Choose one with the smallest possible domain.*

For $H = C_3, G = C_6$, $\theta(n) = 0$.

For $H = C_3, G = S_3$, $\theta(n) =$.

For $H = V_4, G = A_4$, $\theta(n) =$

e.) *What map θ' from $\frac{G}{H}$ would satisfy the equation $Im(q) = Ker(\theta')$? Choose one with the smallest possible domain.*

For $H = C_3, G = C_6$, $\theta'(n) = 0$.

For $H = C_3, G = S_3$, $\theta'(n) =$.

For $H = V_4, G = A_4$, $\theta'(n) =$.

f.) *Add the two maps θ and θ' to your illustration.*

The new hain of four homomorphisms is called a short exact sequence. It is one way to use homomorphisms to illustrate quotients, and show its a connection between embeddings and quotient maps.

8.15

Figure 5.8 on page 69 shows the patern in Cayley diagrams distinguishing abelian and non-abelian groups, the visualization of the equation $ab = ba$.

a.) *Use algebra to show that the equation $aba^{-1}b^{-1} = e$ is equivalent to the original.*

$aba^{-1}b^{-1} = e$

$aba^{-1}b^{-1}b = eb$

$aba^{-1}e = b$

$aba^{-1} = b$

$aba^{-1}a = ba$

$abe = ba$

$ab = ba$

b.) *Use algebra to show that it is also equivalent to the equation* $ab(ba)^{-1} = e$.

$ab(ba)^{-1} = e$

$aba^{-1}b^{-1} = e$

$aba^{-1}b^{-1}b = eb$

$aba^{-1}e = b$

$aba^{-1} = b$

$aba^{-1}a = ba$

$abe = ba$

$ab = ba$

c.) *Create an illustration of what* $aba^{-1}b^{-1} \neq e$ *looks like in a Cayley diagram.*

Based on Exercise 8.15, a group G *containing an element* $ab(ba)^{-1}$ *that is not the identity element* e *cannot be abelian. Such elements are called commutators. We wish to form the cummutator subgroup, generated by the set of all commutators. Then we will divide* G *by it to eliminate all the elements that keep* G *from being abelian, and an abelian group will result.*

8.16

Explain why the commutator subgroup must be a normal subgroup.

Let C be the commutator subgroup for some group G and let $g, h \in G$ such that $gC, hC \in G/C$.

The commutator $ghg^{-1}h^{-1} \in C$, since C is the commutator subgroup.

Thus $ghg^{-1}h^{-1}C = C$.

$ghg^{-1}h^{-1}hC = Ch$

$ghg^{-1}eC = Ch$

$ghg^{-1}gC = Chg$

$gheC = Chg$

$ghC = Chg$

Which implies that for any elements in G, the left coset of C equals the right coset of C and, thus, $C \triangleleft G$.

8.17

The abelianization of a group G is the quotient of G by its commutator subgroup.

a.)*Compute the abelianization of S_3.*

We know that the commutator subgroup must be a normal subgroup.

The only normal subgroups of S_3 are the trivial, A_3 and S_3.

$Com_{S_3} = \langle aba^{-1}b^{-1} | a, b \in S_3 \rangle$

Therefore the commutator subgroup must be $Com_{S_3} = A_3$.

$S_3/Com_{S_3} = S_3/A_3 = A_3$

b.)*Compute the abelianization of A_4.*

The only normal subgroups of A_4 are the trivial, $\langle (0\ 1)(2\ 3), (0\ 3)(1\ 2) \rangle$, and the non-proper. Therefore, these are the only possibilities for the commutator group.

We know the commutator subgroup is the set of element that commute with eachother, so the non-proper subgroup can't be the commutator subgroup. Therefore, the $Com_{A_4} = \langle (0\ 1)(2\ 3), (0\ 3)(1\ 2) \rangle$.

Thus $A_4/Com_{A_4} \cong C_3$

c.)*Compute the abelianization of D_5. What does it have in common with the abelianization of D_3 from part (a)?*

For D_5 ,the only normal subgroups are the trivial, $\langle r \rangle$, and the non-proper. So, we only need to check those subgroups.

$rrr^{-1}r^{-1} = r^2r^{-2} = e$

$r^2r^2r^{-2}r^{-2} = r^4r^{-4} = e$

$r^3r^3r^{-3}r^3 = r^6r^{-6} = e$

$rfrfr^{-1}(frf)^{-1} = rr^{-1}r^{-1}r^{-1} = r^{-2} = r^3 \in \langle r \rangle$

d.) *The group D_2 is isomorphic to V_4, which is abelian. What is its abelianization?*

Because $D_2 \cong V_4$ is abelian, the commutator subgroup consists only of the identity element and thus the abelianization returns the group itself.

e.) *Compute the abelianization of the groups D_4 and D_6.*

ForD_4, the normal subgroups are the trivial, $\langle r^2 \rangle$,$\langle r \rangle$, $\langle r^2, f \rangle$, $\langle r^2, fr \rangle$, and the non-proper subgroup, so we only need to check those subgroups.

$fff^{-1}f^{-1} = ff^{-1} = e$

$rrr^{-1}r^{-1} = rr^{-1} = e$

$r^2r^2r^{-2}r^{-2} = r^2r^{-2} = e$

$rfr^{-1}f^{-1} = fr^{-1}r^3f = fr^2f = r^2ff = r^2$

$rfrr^{-1}(fr)^{-1} = rfrr^{-1}r^{-1}f = rfr^{-1}f = rfr^3f = fr^{-1}r^3f = fr^2f = ffr^2 = r^2$

Therefore $Com_{D_4} = \langle r^2 \rangle$ and $D_4/\langle r^2 \rangle \cong C_2 \times C_2$

For D_6 , the only normal subgroups are the trivial, $\langle r^2 \rangle$, $\langle r \rangle$, $\langle r^2, f \rangle$, $\langle r^2, fr \rangle$ and the non-proper subgroup, so we need to check the same elements.

$fff^{-1}f^{-1} = ff^{-1} = e$

$rrr^{-1}r^{-1} = rr^{-1} = e$

$r^2r^2r^{-2}r^{-2} = r^2r^{-2} = e$

$r^3r^3r^{-3}r^{-3} = r^6r^{-6} = e$

$rfr^{-1}f^{-1} = fr^{-1}r^3f = fr^2f = r^2ff = r^2$

$rfrr^{-1}(fr)^{-1} = rfrr^{-1}r^{-1}f = rfr^{-1}f = rfr^3f = fr^{-1}r^3f = fr^2f = ffr^2 = r^2$

Therefore $Com_{D_6} = \langle r^2 \rangle$ and $D_6/\langle r^2 \rangle \cong C_2 \times C_2$

f.) *What general conclusion do you draw about the abelianization of dihedral groups?*

For D_n with $2n$ elements, we know that $r^n = e$, $f^2 = e$ and $fr = r^{-1}f$

$fff^{-1}f^{-1} = fef^{-1} = ff^{-1} = e$

$rrr^{-1}r^{-1} = rer^{-1} = rr^{-1} = e$

$r^l fr^k r^{-l}(fr)^{-k} = r^l fr^k r^{-l} r^{-k} f = f^{2l-2k} = (r^2)^{l-k}$

$fr^l fr^k (fr)^{-l}(fr)^{-k} = fr^l fr^k (r^{-l}f)(r^{-k}f) = r^{k-l}r^{k-l} = (r^2)^{k-l}$

Therefore

$$Com_{D_n} = \begin{cases} \langle r^2 \rangle & \text{if } n \text{ is even, with order} \frac{n}{2} \\ \langle r \rangle, & \text{if } n \text{ is odd, with order} n \end{cases}$$

By Lagrange's Theorem,

$$|D_n/Com_{D_n}| = \begin{cases} \frac{2n}{\frac{n}{2}} = 4 & \text{if } n \text{ is even} \\ \frac{2n}{n} = 2 & \text{if } n \text{is odd} \end{cases}$$

Therefore, when n is odd, there are there are two distinct left cosets of Com_{D_n}, $eCom_{D_n}$ and $fCom_{D_n}$ and $D_n/Com_{D_n} \cong C_2$

When n is even, there are four distinct left cosets of Com_{D_n}, $eCom_{D_n}$, $rCom_{D_n}$, $fCom_{D_n}$, $frCom_{D_n}$ and $D_n/Com_{D_n} \cong C_2 \times C_2$.

8.36

Prove that $A \times B \cong B \times A$. Give the formula for the isomorphism.

Let $\phi((a,b)) : A \times B \to B \times A$ such that $\phi((a,b)) = (b,a)$. We will first show that this map is a homomorphism.

Let $(a,b), (c,d) \in A \times B$.

$\phi((a,b))\phi((c,d)) = (b,a)(d,c) = (bd,ac) = \phi((ac,bd)) = \phi((a,b))\phi((c,d))$.

Thus ϕ is a homomorphism. Since ϕ maps every element in $A \times B$ to every element in $B \times A$, we know that it is one-to-one and onto, and thus a bijection.

Therefore, ϕ is an isomorphism and $A \times B \cong B \times A$.