Modern Algebra
Homework 8b
Chapter 8
Read 8.3
Complete 8.13, 8.14, 8.20, 8.22, 8.23, 8.40
Proof

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8.13

For any group G and any number n we can create a homomorphism that raises every element to the n^{th} power, $\phi : G \rightarrow G$ by $\phi(g) = g^n$. (In an additive group, we would write ng instead of g^n. Thus this ϕ is like the function in exercise 8.5, but it works for any group G.

a.) *What is the kernel of this homomorphism?*

The $Ker(\phi) = \{g \in G | g^n = e\}$.

b.) *When we compute $\frac{G}{Ker(\phi)}$, do we get a subgroup of G?*

No, the quotient group is a group of cosets that is isomorphic to the subgroup $Ker(\phi)$ of G.

c.)*Is $\frac{G}{H}$ always isomorphic to a subgroup of G (for any G and $H \lhd G$)?*

No, it isn't always isomorphic. An example would be $H = \langle -1 \rangle$. $Q_4/H \cong V_4$, but there is no embedding from V_4 to Q_4.

8.14

For any group, G, consider the homomorphism $\phi : G \to G$ by $\phi(g) = g^{-1}$. What are its image and kernel? What more can you say about it?

The $Im(\phi) = G$, since every element in G has an inverse and every element gets mapped to its inverse. The $Ker(\phi)$ is simply the identity element, since the only element which has the identity as its inverse is the identity itself. We can say that the homomorphism is in fact an isomorphism ϕ is bijective.

8.20

For each number given below, find the smallest nonnegative integer to which it is congruent mod 12.

a.)*15*

$15 \equiv_{12} b$ implies $12|(15 - b)$. The smallest possible nonegative integer is $b = 3$.

b.)*30*

$30 \equiv_{12} b$ implies $12|(30 - b)$. The smallest possible nonnegative integer is $b = 6$, since $30 = 2 * 15$ and $15 \equiv_{12} 3$.

c.)*529* $529 \equiv_{12} b$ implies $12|(529 - b)$. The smallest possible nonnegative integer is $b = 1$, since $44 * 12 + 1 = 529$.

d.)-*9* $-9 \equiv_{12} b$ implies $12|(-9 - b)$ and $-9 - b = 12 * k$, for some $k \in \mathbb{Z}$. Let $k = -1$, then $b = 3$, which is the smallest nonnegative integer possible.

e.)-*182* $-182 \equiv_{12} b$ implies $12|(-182 - b)$ and $-182 - b = 12 * k$ for some $k \in \mathbb{Z}$. Let $k = -16$, then $b = 10$, which is the smallest nonnegative integer possible.

8.22

For each of the following statements, determine whether it is true or false.

a.)*If $a \equiv_6 b$, then $a \equiv_{12} b$.*

$a \equiv_6 b$ means that $6|(a - b)$, which does not necessarily imply that $12|(a - b)$, since $12 > 6$. Therefore, the statement is false.

b.)*If* $a \equiv_6 b$*, then* $a \equiv_3 b$*.*

$a \equiv_6 b$ means that $6|(a-b)$, which implies that $3|(a-b)$ since $3|6$. Therefore, the statment is true.

c.)*If* $a \equiv_6 b$*, then* $a \equiv_5 b$*.*

If $a \equiv_6 b$, then $6|(a-b)$, which does not necessarily imply that $5|(a-b)$, since $5 \nmid 6$.

d.)*If* $a \equiv_{12} b$*, then* $a \equiv_2 b$*.* If $a \equiv_6 b$, then $6|(a-b)$, which implies that $2|(a-b)$, since $2|6$.

8.23

Let p *be prime and consider the group* $C_p \times C_p$*.*

a.)*Let* (a, b) *be any non-identity element in the group. What is its order? How do you know?*

Since C_p is cyclic, the order of any $a \in C_p$, $|a| = p-a$. Similarilly, $|b| = p-b$.

Since $C_p \times C_p$ is a direct product group, the order of (a, b), $|(a, b)|$ is the least common multiple of the orders of the individual elements, that is $lcm\{|a|, |b|\}$.

The reasoning behind this is that in a direct product group the elements act independtly, so to return to the identity element, we must first return to the identy of one factor, then the other.

b.)*If* (a, b) *and* (c, d) *are both elements of* $C_p \times C_p$ *and neither one is in the orbit of the other, then do their orbits overlap at all?*

No, they don't overlap in the sense that every element of $C_p \times C_p$ belongs to exactly one orbit. They do overlap in the sense that the elements of the factors may appear in seperate orbits, but never in the same element of $C_p \times C_p$.

c.)*How many different orbits are there in* $C_p \times C_p$*?*

There will be $p + 1$ orbits in $C_p \times C_p$.

d.)*What does a cycle graph of* $C_p \times C_p$ *look like?*

A flower where each 'petal' contains p elements.

8.40

Recall that the group $\mathbb{Q}$ (under addition) and the group $\mathbb{Q}^*$ (under multiplication) introduced in Exercise 4.33. Show that $\mathbb{Q} \times C_2 \cong \mathbb{Q}*$ by specifying the isomorphism and explaining why the function you give is indeed an isomorphism.

Let $\phi : \mathbb{Q} \times C_2 \to \mathbb{Q}^*$ be an isomorphism.

For any $\frac{p}{q} \in \mathbb{Q}$, $\phi(\frac{p}{q} = \phi(\frac{1}{q} \cdots \frac{1}{q}) = \phi(\frac{1}{q}) + \cdots + \phi(\frac{1}{q}) = p\phi(\frac{1}{q})$.

$\phi(1) = \phi(\frac{1}{q}) + \cdots \phi(1) = q\phi(\frac{1}{q}) = q\phi(\frac{1}{q})$.

Therefore, $\phi(\frac{1}{q}) = \frac{\phi(1)}{q}$.

This implies that $\phi(\frac{p}{q}) = \frac{p}{q}\phi(1)$.

Therefore $\phi(\mathbb{Q}) \times C_2) = \mathbb{Q} \times C_2\phi(1)$.

Therefore, $\mathbb{Q} \times C_2 \cong \mathbb{Q}*$, for the isomorphism $\phi : \mathbb{Q} \times C_2 \to \mathbb{Q}^*$.

Proof 1

a.)*Prove that if $H < G$, then $H \cong gHg^{-1}$ for any $g \in G$. (Recall that we showed in class that gHg^{-1} is always a subgroup of G.)*

Define the map $\phi : H \to gHg^{-1}$ where $\pi(h) = ghg^{-1}$ for all $g \in G$ and $h \in H$.

Let $a, b \in H$.

$\phi(ab) = gabg^{-1} = gaebg^{-1} = gag^{-1}gbg^{-1} = \phi(a)\phi(b)$.

Therefore, the map ϕ is a homomorphism.

Let $a, b \in H$ such that $\phi(a) = \phi(b)$.

$gag^{-1} = \phi(a) = \phi(b) = gbg^{-1}$

Which implies, $gag^{-1} = gbg^{-1}$

$g^{-1}gag^{-1} = g^{-1}gbg^{-1}$

Therefore, $ag^{-1} = bg^{-1}$

$ag^{-1}g = bg^{-1}g$

Which implies, $a = b$ and the homomorphism ϕ is one-to-one.

Let $c \in gHg^{-1}$, then $c = ghg^{-1}$ for some $h \in H$, by definition of gHg^{-1}.

Since $h \in H$, $\phi(h) = ghg^{-1} = c$.

Therefore, ϕ is onto.

Since the homomorphism ϕ is both one-to-one and onto, ϕ is an isomorphism.

Therefore, for any $g \in G$, $H \cong gHg^{-1}$.

b.) *Use the proof from part (a) to show that in any group $|xy| = |yx|$.*

Let $x, y, g \in G$ and $n, m \in \mathbb{N}$ such that $|x| = n$ and $|gxg^{-1}| = m$.

We know that, by definition, the order of an element is the smallest integer such that $x^n = e$.

$(gxg^{-1})(gxg^{-1}) = gxexg^{-1} = gx^2g^{-1}$

Therefore, $(gxg^{-1})^n = gxg^{-1}gxg^{-1} \cdots gxg^{-1} = gx^ng^{-1}$

Yet, we know from the above definition that $x^n = e$.

Therefore $gx^ng^{-1} = geg^{-1} = gg^{-1} = e$.

Thus, by the definition of order, since $(gxg^{-1})^n = e$, we know that $|gxg^{-1}| = n = |x|$.

Let $|xy| = k$ for some $k \in \mathbb{N}$. Then,

$(xy)^n = xy \cdots xy = e$.

$xy \cdots xyy^{-1} = y^{-1}$

$xy \cdots xy * x = y^{-1}$

$yx * yx \cdots yx = yx^n = yy^{-1} = e$

Therefore, by the definition of order, $|yx| = n = |xy|$.

Proof 2

Prove that if A and B are normal subgroups of G and $AB = G$, then $G/(A \cap B) \cong (G/A) \times (G/B)$. [Hint: Construct a homomorphism $\phi : AB \to (G/A) \times (G/B)$ that has a kernel $A \cap B$, then apply the FHT.]

Let the map $\phi : G \to G/A \times G/B$ such that $\phi(g) = (gA, gB)$ for all $g \in G$.

Let $a, b \in G$, then $\phi(ab) = (abA, abB) = (aAbA, aBbB) = (aA, aB)(bA, bB)$ $= \phi(a)\phi(b)$

Therefore, ϕ is an homomorphism.

$g \in Ker(\phi)$ if and only if $\phi(g) = (gA, gB) = (eA, eB)$.

That implies $gA = eA$ and $gB = eB$.

This implies that $g \in A \cap B$ and $Ker(\phi) = A \cap B$.

Let $(aA, bB) \in (G/A) \times (G/B)$.

Since $G = AB$, there exists $b \in G = AB$.

This implies there exists $m_1 \in A, n_1 \in B$ such that $b = m_1 n_1$.

Therefore $bB = m_1 n_1 B = m_1 B$.

Since $G = AB$, we know that AB is a subgroup of G, in particular the non-proper subgroup. Therefore $AB = BA$.

This implies there exists $a \in BA$ such that there exists $m_2 \in A, n_2 \in B$ where $a = n_2 m_2$.

Therefore $aA = n_2 m_2 A = n_2 A$.

Since $B \lhd AB$, we know that $m_1^{-1} n_2 m_1 \in B$.

$\phi(n_2 m_1) = (n_2 m_1 A, n_2 m_1 B) = (n_2 A, m_1 m_1^{-1} n_2 m_1 B) = (n_2 A, m_1 B) = (aA, bB)$.

Therefore, ϕ is onto.

Let $a, b \in G$ such that $\phi(a) = \phi(b)$.

$(aA, aB) = \phi(a) = \phi(b) = (bA, bB)$

$(aA, aB) = (bA, bB)$ which implies $a = b$.

Therefore ϕ is one-to-one.

Since ϕ is one-to-one and onto, ϕ is a isomorphism.

Since every element in $G = AB$ is in $(G/A) \times (G/B)$, the $Im(\phi) = (G/A) \times (G/B)$.

By the Fundamental Homomorphism Theorem, $Im(\phi) \cong G/Ker(\phi)$ where $Ker(\phi) = A \cap B$.