Modern Algebra
Homework 8c
Chapter 8
Read 8.4-8.5
Complete 8.39a, 8.41, 8.42, 8.43, 8.50
Proof

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8.39a

Explain why Q_4 is not isomorphic to any member of the families of groups we met in Chapter 5.

Q_4 is not isomorphic to any members of the five families because it is non-abelian, but every subgroups is normal.

8.41

The group U_n contains the numbers between 1 and n that are relatively prime to n, with the operation of multiplication mod n. So, for example $U_8 = \{1, 3, 5, 7\}$, and has the following multiplication table.

a.) *To what more familiar group is U_8 isomorphic?*

It is isomorphic to V_4.

b.) *What are the orders of the groups U_n for $n \leq 10$?*

$U_1 = \{1\}$
$|U_1| = 1$

$U_2 = \{1\}$
$|U_2| = 1$

$U_3 = \{1, 2\}$
$|U_3| = 2$

$U_4 = \{1, 2, 3\}$
$|U_4| = 3$

$U_5 = \{1, 2, 3\}$
$|U - 5| = 3$

$U_6 = \{1, 5\}$
$|U_6| = 2$

$U_7 = \{1, 2, 3, 5\}$
$|U_7| = 4$

$U_8 = \{1, 3, 5, 7\}$
$|U_8| = 4$

$U_9 = \{1, 2, 5, 7\}$
$|U_9| = 4$

$U_{10} = \{1, 3, 7\}$
$|U_{10}| = 3$

c.) *What is the relationship between U_5 and U_{10}?*

They are isomorphic to each other, where $\phi : U_{10} \to U_5$ is defined by $\phi(a) = a mod 5$ for any $a \in U_{10}$.

d.) *Examine U_p for the first few primes p. What conjecture do you make about U_p for any prime?*

U_p is always cyclic when p is prime.

e.) *All the groups U_n belong in which of the families of groups we met in Chapter 5?*

U_n all belong to the family of Abelian groups since all subgroups of U_n are normal.

The family of groups U_n has several interesting properties. For instance, every finite abelian group is isomorphic to a subgroup of some U_n.

8.42

For each part below, consider the group generated by the two matrices shown, using matrix multiplication as the binary operator. To what common group is it isomorphic? What is the isomorphism?

a.)Let $x = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, Let $y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$xy = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = z$

$yx = z$

$xx = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = t$

$yy = t$

$yt = ty = y$

$xt = tx = x$

$zt = tz = z$

$zx = xz = y$

$yz = zy = x$

$zz = t$

The group is isomorphic to V_4. The isomorphism is $\phi(x) = h$, $\phi(y) = v$.

b.)Let $x = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, Let $y = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ where $i = \sqrt{-1}$.

$xy = \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix} = z$

$yx = z$

$xx = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = t$

$yy = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = v$

$zz = t$

$tt = tx = xt = ty = yt = tz = zt = tv = vt$

$xv = vx = t$

$zv = vz = z$

$xz = zx = y$

$yz = zy = t$

$zv = vz = y$

$vv = t$

This group is isomorphic to $C_4 \times C_2$.

c.)Let $x = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, Let $y = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$xy = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = z$

$yx = y$

$xz = y$

$zx = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = b$

$xx = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = t$, therefore t is the identity.

$yy = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = a$

$zz = b$

$ab = a$

$ba = a$

$at = a$

$ta = a$

$bt = b$

$tb = b$

$aa = a$

$bb = a$

$$xb = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = c$$

$bx = z$

$yb = c$

$by = z$

$cx = y$

$xc = b$

$cy = y$

$yc = a$

$zc = a$

$cz = a$

$ca = a$

$ac = a$

$cb = a$

$bc = b$

The group is isomorphic to D_3 where $\phi(x) = f$ and $\phi(y) = f$.

8.43

This problem deals with the special case when H and K are both normal subgroups. Consider the homomorphism $\theta : H \times K \to G$ by $\theta(h,k) = hk$, which takes pairs of elements from $H \times K$ and multiplies them in G. Notice that $Im(\theta) = HK$.

a.)*If H and K intersect only at the identity element, explain why θ is an isomorphism (and thus $H \times K \cong HK$).*

Let $h \in H, k \in K$ where $\theta(h,k) = \theta(k,h)$.

$\theta(h,k) = hk = kh = \theta(k,h)$

$hkh^{-1} = khh^{-1} = k$

$hkh^{-1}k^{-1} = kk^{-1} = e$

$hkh^{-1}k^{-1} = h(kh^{-1}k^{-1} \in H$, since H is normal.

$hkh^{-1}k^{-1} = (hkh^{-1})k^{-1} \in K$, since K is normal.

Therefore $hkh^{-1}k^{-1} \in H \cap K = \{e\}$.

Let $h_1, h_2 \in H, k_1, k_2 \in K$.

$\theta((h_1,k_1)(h_2,k_2)) = \theta(h_1h_2, k_1k_2) = h_1h_2k_1k_2 = (h_1k_1)(h_2k_2) = \theta(h_1,k_1)\theta(h_2,k_2)$.

Therefore, θ is a homomorphism.

If $(h,k) \in Ker(\theta)$, then $hk = e$ and $h = k^{-1}$.

This implies that $h \in H \cap K = \{e\}$, which implies $h = eandk^{-1} = e$, therefore $k = e$.

Therefore the $Ker(\theta) = \{e\}$ and, by the fundamental homomorphism theorem, θ is an isomorphism (and thus $H \times K \cong HK$).

b.)*Is the reverse also true? That is, if $H \times K \cong HK$, must H and K only overlap at e?*

Yes, it is true since by the homomorphism theorem, the $Im(\theta) \cong \frac{H \times K}{Ker(\theta)}$. This can only be true if $H \cap K = \{e\}$.

8.50

If an infinite abelian group G is generated by $g_1, \cdots, g_n$ only some of which have finite order, then how might we write G as a cross product of cyclic groups? (Not necessary to prove answer, simply make a reasonable conjecture).

We could expand the Fundamental Theorem of Abelian Groups which says that every abelian group is isomorphic to a direct product of cyclic groups.

I would then define an automorphism that is an isomorphism.

I would then use that isomorphism to show that G isomorphic to $C - 2 \times C_2$, C_3, or C_2.