LINEAR PROGRAMMING

Homework 7

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1. *Suppose you were asked to solve the following LP:*

$$\begin{aligned} \max\ & -x_1 - x_2 - x_3 - x_4 \\ \text{s.t. } & 3x_1 = 78 \\ & 4x_2 = 40 \\ & 4x_3 = 30 \\ & 6x_4 = 30 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Of course, there was nothing to do here: the optimal basis is $B = [1, 2, 3, 4]$ *and with* $x_B^* = [26, 10, 7.5, 5]^T$.

(a) *You realize that you copied the problem down wrong, and you forgot the variable* x_5 *(which has to be* ≥ 0*). The objective coefficient is* $c_5 = -1$*, and the column for* x_5 *in the constrain matrix is* $A_5^T = \begin{bmatrix} 2 & 0 & 2 & 0 \end{bmatrix}$.
Use the revised simplex method to find the optimal solution for the problem starting from the current basic solution.

$$A_B = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix} c_B = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$

First, we must determine the dual optimal solution.

$$A_B^T y = C_B$$

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{4} \\ -\frac{1}{4} \\ -\frac{1}{6} \end{bmatrix}$$

Now we must perform the minimum ratio test to determine the entering variable.

k	c_k		$y^T A_K$
5	-1	$\geq$	$-\frac{7}{6}$

Therefore x_5 will enter the basis.

Now we must update our matrices accordingly.

$$A_B d = A_K$$

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

Now that we have obtained d we must determine the leaving variable.

$$t = \min\{\frac{x_B^*}{d_B}\}$$

$$t = \min\{\begin{bmatrix} 39 \\ DNE \\ 15 \\ DNE \end{bmatrix} = 15$$

Therefore, x_3 leaves the basis.

Now we must update our solution x_B^*.

$$x_1 = 26 - \frac{2}{3} * 15 = 16$$
$$x_2 = 10 - 0 * 15 = 10$$
$$x_4 = 5 - 0 * 15 = 5$$
$$x_5 = 15$$

(b) *No way! You now see that there is another variable you forgot. The variable you forgot is x_6 (which has to be ≥ 0). The objective coefficient is $c_6 = -1$, and the column for x_6 in the constraint matrix is $A_6^T = \begin{bmatrix} 2 & 1 & 0 & 1 \end{bmatrix}$.*
Use the revised simplex method to find the optimal solution for the problem starting from the basic solution you found in part (a).

$$A_B = \begin{bmatrix} 3 & 0 & 2 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix} c_B = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} x_B^* = \begin{bmatrix} 16 \\ 10 \\ 5 \\ 15 \end{bmatrix} B = \begin{bmatrix} 1 & 2 & 4 & 5 \end{bmatrix}$$

First, we must determine the dual optimal solution.

$$A_B^T y = C_B$$

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{4} \\ -\frac{1}{6} \\ -\frac{1}{6} \end{bmatrix}$$

Now we must perform the minimum ratio test to determine the entering variable.

k	c_k		$y^T A_K$
6	-1	$\geq$	$-\frac{13}{12}$

Therefore x_6 will enter the basis.
Now we must update our matrices accordingly.

$$A_B d = A_K$$

$$\begin{bmatrix} 3 & 0 & 2 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_4 \\ d_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} d_1 \\ d_2 \\ d_4 \\ d_5 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{4} \\ 0 \\ \frac{1}{6} \end{bmatrix}$$

Now that we have obtained d we must determine the leaving variable.

$$t = \min\{\frac{x_B^*}{d_B}\}$$

$$t = \min\{\begin{bmatrix} 24 \\ 40 \\ DNE \\ 90 \end{bmatrix} = 24$$

Therefore, x_1 leaves the basis.
Now we must update our solution x_B^*.

$$x_2 = 10 - \frac{1}{4} * 24 = 4$$

$$x_4 = 5 - 0 * 24 = 5$$

$$x_5 = 15 - \frac{1}{6}24 = 11$$

$$x_6 = 24$$

(c) *Suppose there is yet another variable you forgot. Call this variable x_7 (which has to be ≥ 0). The objective coefficient is $c_7 = -1$, and the the column for x_7 in the constraint matrix is $A_7 = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix}$. Give the conditions on the values $a_1, \ldots, a_4$ that guarantee that the solution from part (b) stays optimal, even after adding x_7*

$$A_B = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 6 \end{bmatrix} c_B = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} x_B^* = \begin{bmatrix} 4 \\ 5 \\ 11 \\ 24 \end{bmatrix} B = \begin{bmatrix} 2 & 4 & 5 & 6 \end{bmatrix}$$

First, we must determine the dual optimal solution.

$$A_B^T y = C_B$$

$$\begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 4 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -\frac{7}{24} \\ -\frac{1}{4} \\ -\frac{5}{24} \\ -\frac{1}{6} \end{bmatrix}$$

Now we must perform the minimum ratio test to determine the entering variable.

k	c_k		$y^T A_K$
67	-1	$\geq$	$-\frac{7a_1}{24} - \frac{a_2}{4} - \frac{5a_3}{24} - \frac{a_4}{6}$

If the above inequality holds true, then x_7 would enter the basis. In order to ensure that the solution from part (b) remains optimal, the following inequality must be true:

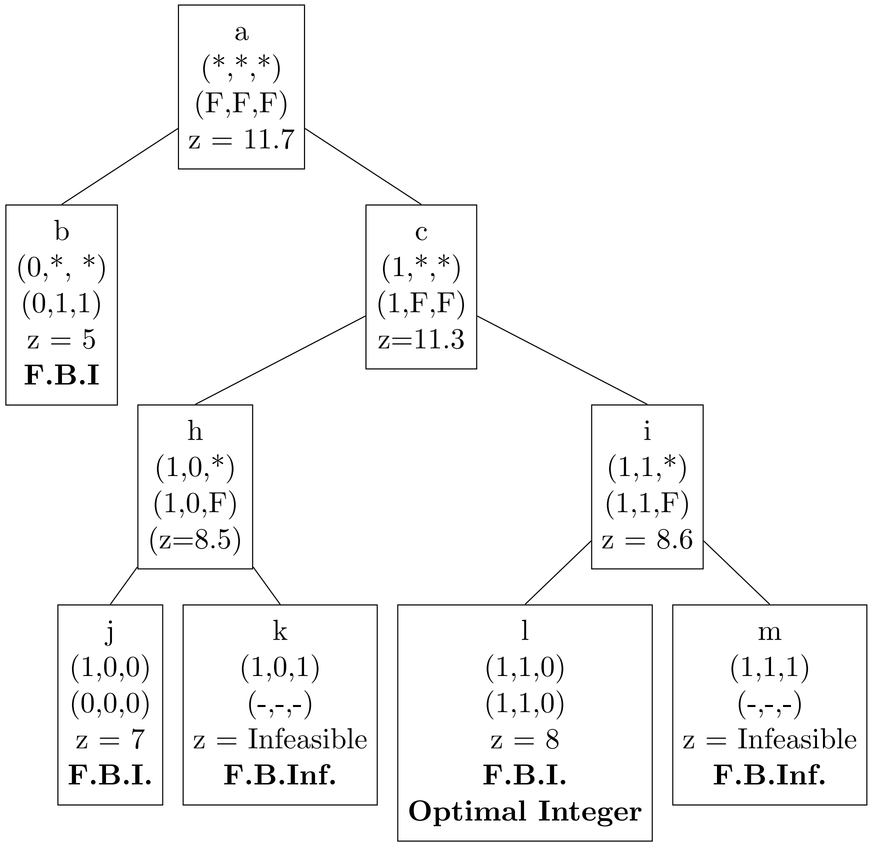
$$\frac{-7a_1}{24} - \frac{a_2}{4} - \frac{5a_3}{24} - \frac{a_4}{6} \geq -1$$

Note: This problem has a very interesting background (including an explanation for why you could have "forgotten" some of the variables when you first solved the problem). We will see this soon in lecture.

2. You wish to apply branch-and-bound to a (maximization) integer program with 0-1 variables. The data at hand has 3 variables. (Clearly, this is just to illustrate the ideas behind algorithms, since for a problem this small, one can easily check all 8 possible solutions, and find the LP relaxation with some of the variables fixed. To simplify things, the table below contains the optimal LP value. For any solution, we let z denote the objective function value; all objective function coefficients are integer. In this table, * indicates no constraint, F indicates that the value is fractional, and - indicates that no feasible solution exists.(Ignore the column labeled heuristic value.) Apply branch-and-bound with this data to obtain the optimal solution showing the tree that is explored, and indicating the order in which the nodes are examined.

Variables Set			LP Optimum				Heuristic Value	**Letter**
x_1	x_2	x_3	x_1	x_2	x_3	z		
*	*	*	F	F	F	11.7	4	a
0	*	*	0	1	1	5		b
1	*	*	1	F	F	11.3	7	c
*	0	*	F	0	F	10.1	7	
*	1	*	F	1	F	9.5	5	
*	*	0	1	1	0	8	8	
*	*	1	F	F	1	10.1	5	
0	*	0	0	1	0	1	1	
0	*	1	0	1	1	5	5	
1	*	0	1	1	0	8	8	
1	*	1	-	-	-	-	-	
*	0	0	1	0	0	7	7	
*	0	1	F	0	1	10	4	
*	1	0	1	1	0	8	8	
*	1	1	F	1	1	9.2	5	
0	0	*	0	0	1	4	4	d
0	1	*	0	1	1	5	5	e
1	0	*	0	1	F	8.5	7	h
1	1	*	1	1	F	8.6	8	i
0	0	0	0	0	0	0	0	
0	0	1	0	0	1	4	4	
0	1	0	0	1	0	1	1	f
0	1	1	0	1	1	5	5	g
1	0	0	0	0	0	7	7	j
1	0	1	-	-	-	-	-	k
1	1	0	1	1	0	8	8	l
1	1	1	-	-	-	-	-	m

The following branch and bound algorithm and corresponding tree were generated in alphabetical order. The first ordered pair is the variables set. The second order pair is the optimal solution. **F.B.I.** means "Fathom by Integrality" and indicates a pruning. **F.B.Inf.** meands "Fathom by Infeasibility" and indicates a pruning. **F.B.B.** means "Fathom by Bounds" and indicates a pruning.



We see from the above branch-and-bound tree that the optimal integer solution occurs at point k and is $\hat{x} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$ with an optimal integer objective value of $z = 8$.

3. *The SAILCO company that you worked for before realized that the fixed costs for starting up their production in any quarter is missing from the model. The cost of 5000, which is incurred if boats are produced in a quarter. If the production in some quarter is 0, then this cost is not incurred in that month. Modify your SAILCO model (or you can use the one posted under HW 3 on Blackboard) to find the optimal producation plan. Hand in your printed model and data file and report the optimal solution (how much to produce each month and how much the total cost is).*

   We see here that even though we added an additional cost in the form of a new variable, the optimal solution did not change. This can be attributed to the fact that the production schedule is still optimal since it was not altered. Therefore the solution is the same as before and SAILCO would perform best if it stuck to the below solution.

   AMPL Code:

```
# SailCo Production Optimization

set QUARTERS = {1,2,3,4} ;
set INVQUARTERS = {0,1,2,3,4};
```

```
param demand {QUARTERS} >= 0;

param prodlim = 40;

param regcost = 400; # cost for the first prodlim boats of the quarter
param overcost = 450; # cost for additional boats
param invcost = 20; # cost for keeping a boat in inventory until the next quarter
param quarterlycost = 5000; # cost for producing goods in any one quarter

param init_inv = 10; # initial inventory

var RegProd {QUARTERS} >= 0, integer; #regular production units
var OverProd {QUARTERS} >= 0, integer; #overtime production units
var Inv {INVQUARTERS} >= 0, integer; #inventory in stock
var x{QUARTERS} binary; #x(i) = 1 if production occurs in quarter i and 0 otherwise

minimize Total_Cost:
regcost * ( sum { t in QUARTERS } RegProd[t] ) + overcost * (sum {t in QUARTERS }
OverProd[t] ) + invcost * (sum {t in QUARTERS} Inv[t]) + (sum {t in QUARTERS} quarterlycos

subject to Initial_Inventory:
Inv[0] = 10;
#starting inventory constraint

subject to End_of_Quarter_Inventory {t in QUARTERS}:
Inv[t] = Inv[t-1] + RegProd[t] + OverProd[t] - demand[t];
#ending inventory is excess production

subject to Max_Reg_Prod {t in QUARTERS}:
RegProd[t] <= 40;
#regular production cannot exceed 40

subject to Quad_Prod_Cost{t in QUARTERS}:
(RegProd[t] + OverProd[t]) - x[t]* demand[t]<= 0;
#ensuring that production costs are assessed.

data;

param demand:=
1 40
2 60
3 75
4 25;
```

AMPL Output:

```
Gurobi 5.6.3: optimal solution; objective 98450
ampl: display RegProd; display OverProd; display x;
RegProd [*] :=
1  40
2  40
3  40
4  25
;

OverProd [*] :=
1   0
2  10
3  35
4   0
;

x [*] :=
1  1
2  1
3  1
4  1
```