Advanced Calculus II
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8.3.2

For $n \in \mathbb{N}$, let $f_n(x) = \frac{x^n}{(1+x^n)}, x \in [0,1]$. Prove that the sequence $\{f_n\}$ does not converge uniformly on $[0,1]$.

Since $x^n < 1 + x^n$, $|f_n(x)| < 1$ for all $x \geq 0$ and f_n is bounded.

For $x \in [0, c]$, with $c \in (0,1)$, $f(x) = \lim_{n \to \infty} f_n(x) = 0$.

For $x = 1$, $f(x) = \lim_{n \to \infty} f_n(x) = \frac{1}{2}$.

Therefore, $f(x) = \begin{cases} 0 & 0 < x < 1 \\ \frac{1}{2} & x = 1 \end{cases}$

Therefore, $f(x)$ is not continuous and the sequence does not converge uniformly.

8.3.5

Let $\{f_n\}$ be a sequence of continuous real-valued functions that converges uniformly to a function f on a set $E \subset \mathbb{R}$. Prove that $\lim_{n \to \infty} f_n(x_n) = f(x)$ for every sequence $\{x_n\} \subset E$, such that $x_n \to x \in E$.

Since f is the uniform limit of continuous functions, f is also continuous by corollary 8.3.2.

Since f is continuous and $x_n \to x$, $f(x_n) \to f(x)$ by theorem 4.1.3.

Let $\epsilon > 0$. Then, there exists $N_1 \in \mathbb{N}$ such that $|f(x_n) - f(x)| < \frac{\epsilon}{2}$ for $n \geq N_1$.

Since f_n converges uniformly to f, there exists $N_2 \in \mathbb{N}$ such that $|f_n(t) - f_n(t)| < \frac{\epsilon}{2}$ for $t \in E$ and $n \geq N_2$.

This implies that for $n \geq N = \max\{N_1, N_2\}$,
$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

Therefore, the $\lim\limits_{n\to\infty} f_n(x_n) = f(x)$ for every sequence $\{x_n\} \subset E$, such that $x_n \to x \in E$

8.3.7

Find a sequence $\{f_n\}$ in $\mathbb{C}[0,1]$ with $||f_n||_u = 1$ such that no subsequence of $\{f_n\}$ converges (in norm) in $\mathbb{C}[0,1]$.

$||f_n||_u = max\{|f(x)| : x \in [0,1]\} = 1$.

Let $||f_n||_\infty = \begin{cases} 1 - nx & 0 < x \leq \frac{1}{n} \\ 0 & \frac{1}{n} < x \leq 1 \end{cases}$

Then $||f_n|| = 1$, but there are no subsequences of $\{f_n\}$ that converge in $\mathbb{C}[0,1]$ since the sequence limit is not continuous.

8.4.3

For each $n \in \mathbb{N}$, let $f_n(x) = \frac{nx}{(1+nx)}, x \in [0,1]$. Show that the sequence $\{f_n\}$ converges pointwise, but not uniformly, to an integrable function f on $[0,1]$, and that $\lim\limits_{n\to\infty} \int\limits_0^1 f_n(x)dx = \int\limits_0^1 dx$.

For $x \in (0,1]$, $f(x) = \lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \frac{nx}{1+nx} = 1$.

For $x = 0, f(0) = \lim\limits_{n\to\infty} f_n(0) = 0$.

Therefore, $f(x) = \begin{cases} 1 & 0 < x \leq 1 \\ 0 & x = 0 \end{cases}$

Therefore $f_n(x)$ is continuous for all $n \in \mathbb{N}, x \in [0,1]$, but $f(x)$ is not continuous.

By the definition of pointwise convergence, since f exists, $f_n(x)$ converges pointwise on $[0,1]$.

Therefore, f_n does not uniformly converge to f on $[0,1]$.

8.4.4

If f is Riemann integrable on $[0,1]$, use the bounded convergence theorem to prove that $\lim_{n\to\infty} \int_0^1 x^n f(x)dx = 0$.

Let $g_n(x) = x^n f(x)$.

Then the pointwise limits, $g(x) = \lim_{n\to\infty} g_n(x) = \begin{cases} 1 & 0 \le x < 1 \\ f(1) & x = 1 \end{cases}$

Therefore, since $g(x)$ is bounded and discontinuous at finitely many points, $x = 1$, $g(x) \in R[0,1]$ by Reimann-Lebesque.

$\int_0^1 g(x) = 0$.

Since x^n is continuous and $f \in R[0,1]$, g_n is also Riemann integrable.

We know that $f(x)$ is bounded since for all $x \in [0,1]$, $|f(x)| \le M$.

Therefore, $|g_n(x)| \le Mx^n \le M$, for all $x \in [0,1]$, $n \in \mathbb{N}$.

This implies that that sequences $\{g_n\}$ is uniformly bounded.

Therefore, by the bounded convergence theorem,
$\lim_{n\to\infty} \int_0^1 g_n(x)dx = \int_0^1 g(x)dx = 0$.

Therefore, if f is Riemann integrable on $[0,1]$, use the bounded convergence theorem to prove that $\lim_{n\to\infty} \int_0^1 x^n f(x)dx = 0$.

8.4.6

Suppose $f : [0,1] \to \mathbb{R}$ is continuous. Prove that $\lim_{n\to\infty} \int_0^1 f(x^n)dx = f(0)$.

Let $g_n(x) = x^n$ for all $x \in [0,1]$.

Then $g_n(x) \geq 0$ and $\int\_0^1 g_n(x)dx = \int_0^1 x^n dx = \frac{1}{n+1}x^{n+1}|_0^1 = \frac{1}{n+1}$.

Therefore, $\lim\limits_{n\to\infty} \int_0^1 g_n(x)dx = 0$.

Since $f(x)$ is continuous and thus Riemann integrable on $[0,1]$, $\lim\limits_{n\to\infty} \int\limits_0^1 f(x)g_n(x)dx =$ $\lim\limits_{n\to\infty} \int_0^1 x^n f(x)dx = \lim\limits_{n\to\infty} \int\limits_0^1 x^n dx \int\limits_0^1 f(x)dx = 0$.

8.5.1

For $n \in \mathbb{N}$, set $f_n(x) = \frac{x^n}{n}, x \in [0,1]$. Prove that the sequence $\{f_n\}$ converges uniformly to $f(x) = 0$ on $[0,1]$: that the sequence $\{f_n'(x)\}$ converges pointwise on $[0,1]$, but that $\{f_n'(1)\}$ does not converge to $f'(1)$.

If $x \in [0,1]$, then $x^n \in [0,1]$ and, thus, $\frac{x^n}{n} \in [0, \frac{1}{n}]$.

This implies that $||f_n|| \leq \frac{1}{n}$.

$f(x) = \lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \frac{x^n}{n} = 0$.

Therefore $\{f_n\}$ converges uniformly to 0 and $f(x)$ is differentiable with $f'(1) = 0$.

$f_n'(x) = x^{n-1}$ for all n.

Therefore, $\lim\limits_{n\to\infty} f_n'(x) = \lim\limits_{n\to\infty} g_n(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$.

Therefore, $f_n'(1) = 1 \neq 0 = f'(1)$ and $\{f_n'(1)\}$ does not converge to $f'(1)$.

8.5.3

Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of real numbers satisfying $\sum k|a_k| < \infty$. Show that the series $\sum\limits_{k=0}^{\infty} a_k x^k$ converges uniformly to a function f on $|x| \leq 1$ and that $f'(x) = \sum\limits_{k=1}^{\infty} k a_k x^{k-1}$ for all $x, |x| \leq 1$.

$\sum k|a_k| < \infty$ implies that $\sum |a_k| < \infty$. Therefore, for all $x \in [-1,1]$ we have that $|x| \leq 1$ and thus $|x|^k \leq 1$.

This implies that $|a_k x^k| = |a_k| * |x|^k \leq |a_k|$ for all $x \in [-1,1]$.

Since $\sum_{k=1}^{\infty} |a_k| < \infty$, we know that $\sum a_k x^k$ converges uniformly to a function f on $[-1, 1]$ by the Weierstrauss M-Test.

$\frac{df}{dx} = \frac{d}{dx} \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} \frac{d}{dx} a_k x^k = \sum_{k=1}^{\infty} k a_k x^{k-1}$, for $|x| \leq 1$ by term-by-term differentiation.

8.5.6a

Show that the following series converges on the indicated interval and that the derivative of the sum can be obtained by term-by-term differentiation of the series: $\sum_{k=1}^{\infty} \frac{1}{(1+kx)^2}$, $x \in (0, \infty)$.

We will use theorem 7.1.2, the comparison test.

Let $S(x) = \sum_{k=1}^{\infty} \frac{1}{(1+kx)^2}$ and $S_n(x) = \sum_{k=1}^{n} \frac{1}{(1+kx)^2}$.

Therefore $S'_n(x) = -2 \sum k = 1^{\infty} \frac{k}{(1+kx)^3}$.

We will use the Weirestrass M-Test and the comparison test to show that the sequences $\{S_n(x)\}$ and $\{S'_n(x)\}$ converge uniformly on $[a, \infty)$ for every $a > 0$.

Since $S(x) = \frac{1}{(1+kx)^2} \leq \frac{1}{k^2}$ for all $x \in \mathbb{R}$, and $\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$, $S(x)$ converges uniformly to a function S on $\mathbb{R}$ by the Weierstrass M-Test.

Since $S_n(x) = \frac{1}{(1+kx)^2} \leq \frac{1}{k^2}$ for all $x \in \mathbb{R}$, and $\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$, $\{S_n(x)\}$ converges uniformly on $\mathbb{R}$ by the Weierstrass M-Test.

Thus, by theorem 8.5.1,$S'(x) = \lim_{n \to \infty} S'_n(x) = \sum_{k=1}^{\infty} -2 \sum k = 1^{\infty} \frac{k}{(1+kx)^3}$, for all $x \in [a, \infty)$.

These results hold for every $a > 0$ and, thus for all $x \in (0, \infty)$.