Modern Algebra
Homework 9a
Chapter 9
Read 9.1-9.3
Complete 9.4, 9.7, 9.9, 9.10
Class Exercises
Proof

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9.4

If $|G| = 28$, what sizes of subgroups does Cauchy's Theorem quarantee exist in G? How does the First Sylow Theorem improve on that guarantee?

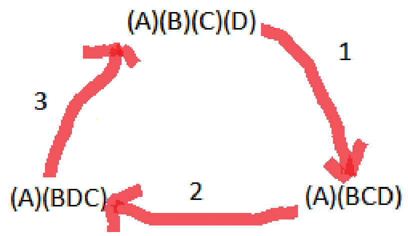
The prime factorization of 28 is $2 * 2 * 7$. Cauchy's Theorem says that if p is a prime number that divides $|G|$, then GQ has an element g of order p and therefore a subgroup $\langle g \rangle$ of order p. Therefore, by Cauchy, there are subgroups of size 2 and size 7.

The First Sylow Theorem improves upon that guarantee by also guaranteeing the existence of a subgroup of size 4, since 2^2 is the highest power that divides $|G|$.

9.7

Consider the action of C_3 on the set $\{A, B, C, D\}$ by the interpretation homomorphism $\phi : C_3 \to Perm(\{A, B, C, D\})$ generated by the following equation: $\phi(1) = (A)(B\ C\ D)$.

a.) *Draw the corresponding action diagram*



b.) *What are the stable elements?*

(A B)
(A C)
(A D)
(B C)
(B D)
(C D)
(A B C)
(A B D)
(A C B)
(A C D)
(A D B)
(A D C)
(A B C D)
(A B D C)
(A C B D)
(A C D B)
(A D B C)
(A D C B)
(A B)(C D)
(A C)(B D)
(A D)(B C)
(A)(B)(C)(D)

c.) *What are the orbits?*

$Orb((A)(B)(C)(D)) = \{(A)(B)(C)(D), (A)(B\ C\ D), (A)(B\ D\ C)\}$

$Orb((A)(B\ C\ D)) = \{(A)(B)(C)(D), (A)(B\ C\ D), (A)(B\ D\ C)\}$

$Orb((A)(B\ D\ C)) = \{(A)(B)(C)(D), (A)(B\ C\ D), (A)(B\ D\ C)\}$

9.9

If C_5 acts on letters $\{A, B, C, D\}$, what will the action diagram be? Why?

Theorem 9.5 states that since C_5 has prime order 5,
$|S| \equiv_5 |\{\text{stable elements in S}\}|$. We also know that the elements of S are either stable or part of an oribt of size 5.

Yet, there are no orbits of size 5 in the $Perm(S)$, therefore the action diagram consists of the identity element and discreet points.

9.10

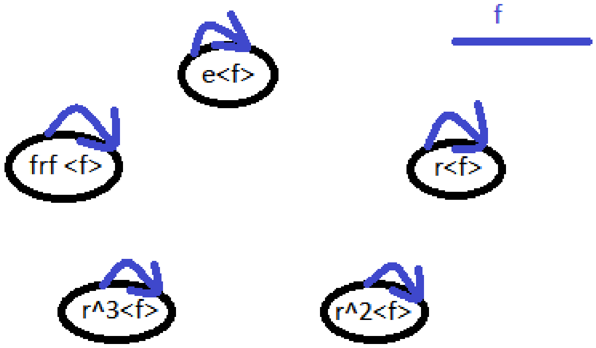
Consider the group action defined at the start of the proof of Theorem 9.9

a.) *Draw an action diagram for the specific example of* $G = D_5$ *and* $H = \langle f \rangle$.

$\phi(h) =$ the permutation that sends a coset gH to the coset hgH.

Elements of $D_5 = \{f, fr, rf, fr^2, r^2f, r, r^2, r^3, frf\}$.

The five left cosets of $\langle f \rangle$ are $e\langle f \rangle, r\langle f \rangle, r^2\langle f \rangle, r^3\langle f \rangle, and frf\langle f \rangle$.



b.) *Verify that for each* $s \in S$, *your diagram satisfies the Orbit-Stabilizer Theorem.*

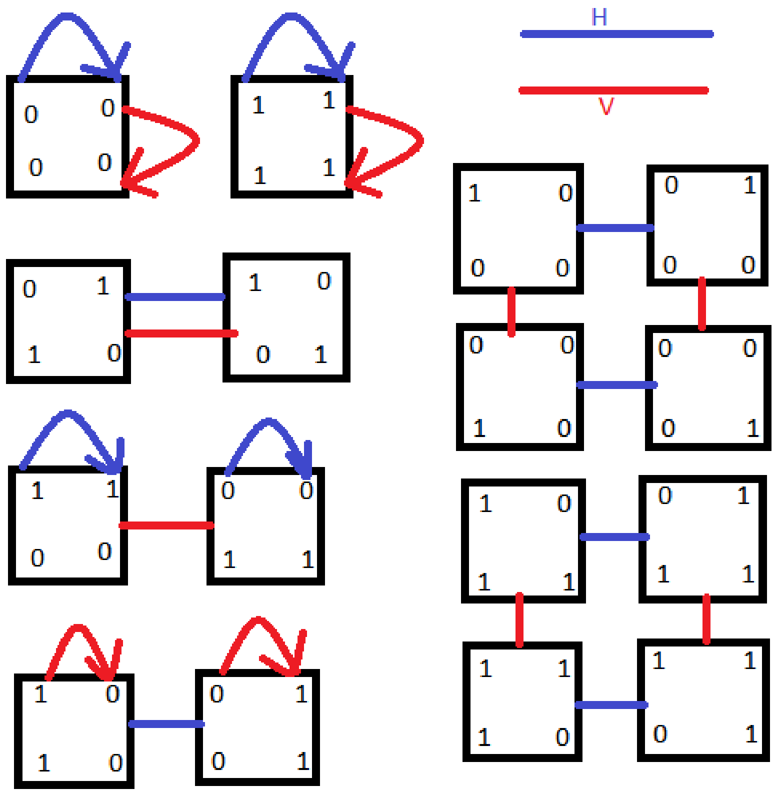
By the orbit stabilizer theorem $|Orb(s)||Stab(s)| = |G|$. This is true for all $s \in S$, since for each $s \in S$, the $|Orb(s)| = 1$ and the $|Stab(s)| = 2$.

$|Orb(s)||Stab(s)| = 1 * 2 = 2 = |\langle f \rangle| = |G|$.

Class Exercise

Repeat the exercise from the class lecture notes for several other groups: Let S be the set of "binary squares". Draw an action diagram for each of the following group actions:

a.)$\phi : V_4 \to Perm(S)$, *where* $\phi(h)$ *reflects each square horizontally, and* $\phi(v)$ *reflects each square vertically.*



For element $(0, 0, 0, 0)$, $Stab(s) = \{e, h, v, hv\}$.

For element $(1, 1, 1, 1)$, $Stab(s) = \{e, h, v, hv\}$.

For element $(0, 1, 0, 1)$, $Stab(s) = \{e, hv\}$.

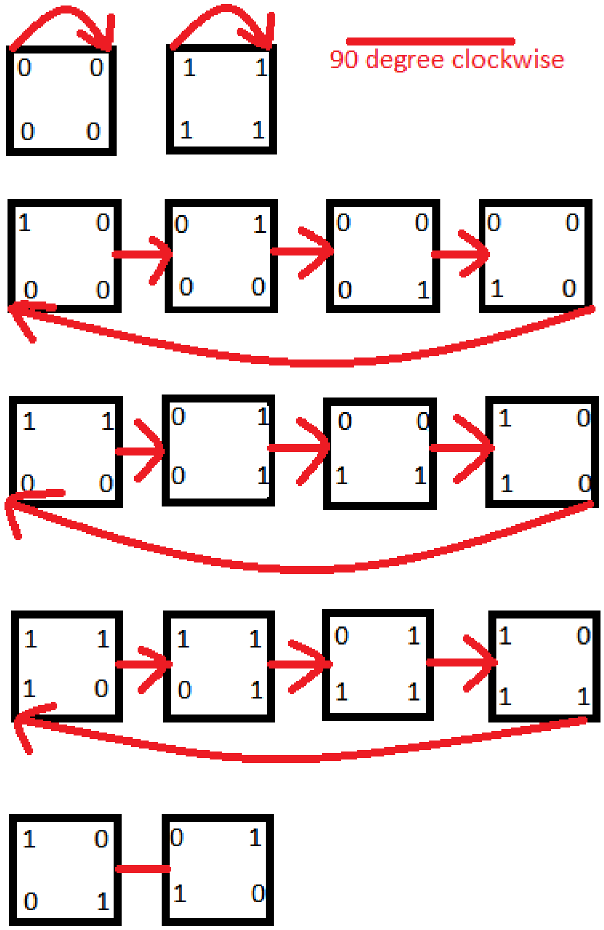
For element $(1, 1, 0, 0)$, $Stab(s) = \{e, h\}$.

For element $(1, 0, 0, 1)$, $Stab(s) = \{e, v\}$.

For element $(1, 0, 0, 0)$, $Stab(s) = \{e\}$.

For element $(1, 0, 11)$, $Stab(s) = \{e\}$.

b.)$\phi : C_4 \to Perm(S)$, *where* $\phi(1)$ *rotates each square* $90°$ *clockwise.*



For element $(0, 0, 0, 0)$, $Stab(s) = \{0, 1, 2, 3\}$.

For element $(1, 1, 1, 1)$, $Stab(s) = \{0, 1, 2, 3\}$.

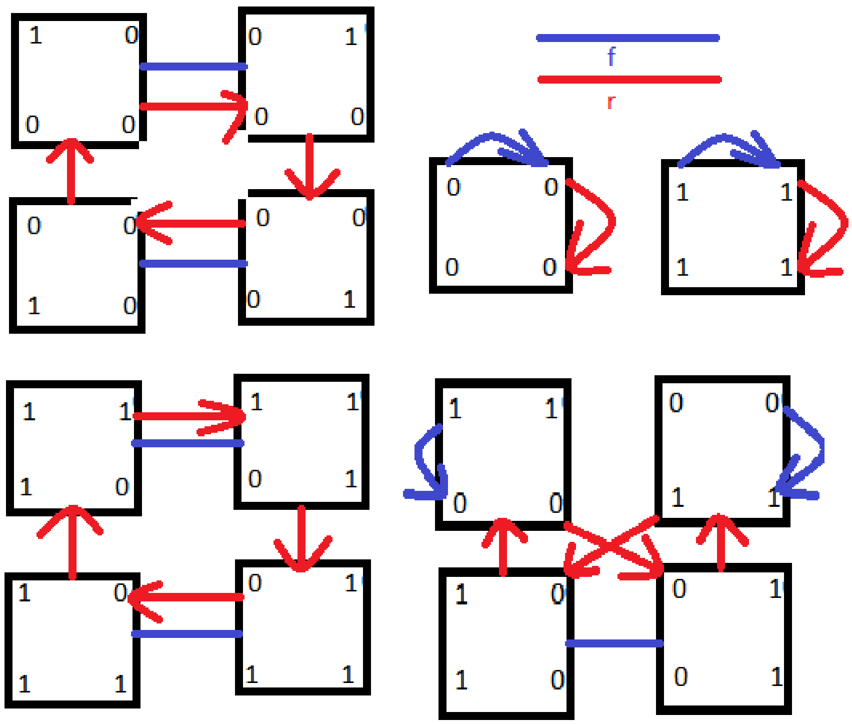
For element $(1, 0, 0, 0)$, $Stab(s) = \{0\}$.

For element $(1, 1, 1, 0)$, $Stab(s) = \{0\}$.

For element $(1, 1, 0, 0)$, $Stab(s) = \{0\}$.

For element $(1, 0, 1, 0)$, $Stab(s) = \{0, 2\}$.

c.)$\phi : D_4 \rightarrow Perm(S)$, *where* $\phi(r)$ *rotates each square* $90°$ *clockwise, and* $\phi(f)$ *reflects each square about a vertical axis.*



For element $(0, 0, 0, 0)$, $Stab(s) = \{e, r, r^2, r^3, f, fr, r^2f, rf\}$.

For element $(1, 1, 1, 1)$, $Stab(s) = \{e, r, r^2, r^3, f, fr, r^2f, rf\}$.

For element $(1, 0, 0, 0)$, $Stab(s) = \{e, rf\}$.

For element $(1, 1, 1, 0)$, $Stab(s) = \{e, rf\}$.

For element $(1, 1, 0, 0)$, $Stab(s) = \{e, rfr\}$.

Additionally, pick an element in each orbit and finds its stabilizer.

Proof 1

Let G be a group of order 15, which acts on a set with 7 elements. Show that the group action has a fixed point.

The size of the orbits will be the prime factors, $1, 3, 5$, or 15.

Since the set has 7 elements, there is not an orbit of 15.

Proof by contradiction: Suppose the group action has no fixed points.

Then, there exists $m, n \in \mathbb{Z}$ such that $3m + 5n = 7$.

This, however, is a condradiction since the smallest nonnegative integer is 1 and $3 + 5 = 8 > 7$.

Therefore, if G is a group of order 15 which acts on a set with 7 elements, the group action must have a fixed point.

Proof 2

Let G act on itself $(i.e., S = G)$ by conjugation.

a.)*Show that the set of fixed points of this action is Z(G), the center of G.*

If the group acts on itself by conjugation, then $\phi : G \times G \to G$ where for all $g, h \in G$, $\phi(g, h) = ghg^{-1}$.

A fixed point of an action is an element $g \in G$ with $|Orb(g)| = 1$.

The $Orb(s) = \{h \in G : \phi(s)^n = h\}$, for some $n \in \mathbb{N}$ with $n \leq |G|\}$.

The $Stab(s) = \{g \in G : \phi(g)s = s\}$.

We know by the orbit stablizer thereom that $|Orb(s)||Stab(s)| = |G|$.

Therefore, the fixed points of this action have $|Stab(s)| = |G|$.

We know that if $g \in S = G$, $xhx^{-1} = h$ if and only if $gh = hg$.

Therefore $Stab(h) = \{g \in G : \phi(g)h = h\}$, which is only true if $gh = hg$.

Therefore, $Stab(h) = Z(G)$.

b.)*Prove that if G is a p-group (i.e., $|G| = p^n$ for some prime p), then $Z(G)$ is nontrivial.*

We have previously shown that the set of fixed points of a conjugation action is the center of G.

If G is a p-group, then $|G| = p^n$ for some $n \in \mathbb{N}$.

By the First Sylow Theorem, there exists a nontrivial normal subgroup N of G.

This implies that $p||N|$.

Since G acts on itself by conjugation, for all $g, h \in G$, $gh = ghg^{-1}$.

We have previously proven that G has a set of fixed points and that set is $Z(G)$.

The identity action, $g = 1$, is always a fixed point in a $p - group$.

Therefore, $g \in N$ and $g \in Z(G)$ and $g \in Z(G) \cap N$.

We know from theorem 9.8 that the order of N and the number of fixed points are congruent mod p, which is congruent to 0 since $p||N|$.

This implies that $|Z(G) \cap N| \geq p$ and the center of $Z(G)$ is nontrivial.

c.)*A group is simple if its only normal subgroups are G and $\{e\}$. Use part (b) to completely classify all simple p-groups.*

Let H be any subgroup of G. This implies that $|H|||G|$.

However, G is a p-group, which implies that $|G| = p^n$ for some $n \in \mathbb{N}$.

If $n = 1$, then $|G|$ is prime, which implies that H is either the trivial or the non-proper subgroup. Thus all p-groups of order p are simple.

If $n > 1$, then $|G|$, then there exists a normal subgroup that is not trivial or non-proper, as shown by part (b).

Therefore, the only p-groups that are simple are p-groups of order p.