Modern Algebra
Homework 9b
Chapter 9
Read 9.1-9.3
Complete 9.21, 9.22, 9.23
Proofs

Megan Bryant

November 20, 2013

First Sylow Theorem

If G is a group and p^n is the highest power of p dividing $|G|$, then there are subgroups of G of every order $1, p, p^2, p^3$, up to p^n. Also, every p-subgroup with fewer than p^n elements is inside one of the larger p-subgroups.

Second Sylow Theorem

Any two Sylow p-subgroups are conjugates.

Third Sylow Theorem

The number n of Sylow p-subgroups of G obeys the following two restrictions: $n||G|$ and $n \cong_p 1$.

9.21

The converse of Lagrange's theorem is true for abelian groups. Any n dividng $|G|$ has a subgroup $H < G$ of order n. Explain why.

The converse of Lagrange's theorem is true for Abelian groups because, by the Fundamental Theorem of Abelian Groups, every finite abelian goup is isomorphic to a direct product of cyclic groups. Cyclic groups can be generated by a single generator and every subgroup of a cyclic group is cyclic. Therefore, any $n||G|$ will have a subgroup of order n if G is Abelian.

9.22

In chapter 10 the simple group A_5 will play an important role; simple groups are those that have no normal subgroups. For each n given below, use Sylow Theory to explain why there can be no simple groups of order n.

a.)*33*
$33 = 3 * 11$. By the First Sylow theorem, any group of order 33 must have a subgroup H of order 3 and K order 11. Every group of prime order is cyclic and, therefore, abelian. Since both of these groups are unique p-subgroups, they are normal subgroups.

We know from Lagrange that $H \cap K = \{e\}$.

Therefore $HK < G$ and $|HK| = |H||K| = 33$.

This implies that $HK = G$.

Therefore, we know that $G \cong H \times K$.

We know that $H \cong C_3$ and $K \cong C_1 1$.

Therefore $G \cong C_3 \times C_1 1$.

Therefore there is only one group of order 33 and that group, as shown, is not simple.

Thus, no groups of order 33 are simple.

b.)*84*
$|G| = 84 = 2^2 * 3 * 7 *$. By the First Sylow theorem, any group of order 84 must have a subgroups of order$2, 2^2, 3$, and 7. Every group of prime order is cyclic and, therefore, abelian.

Let n_2 be the number of Sylow 2-subgroups.

We know that $n_2 = 1, 3, 7$, or 21.

Let n_3 be the number of Sylow 3-subgroups.

We know that $n_3 = 1, 4, 7$, or 28.

The Sylow 7-subgroup is unique and thus normal. Since G has a subgroup that is normal and not the trivial or nonproper, we know that G cannot be simple.

Therefore, there are no simple groups of order 84.

c.) *12*
$12 = 2^2 * 3$. By the First Sylow theorem, any group of order 12 must have a subgroups of order$2, 2^2$ and 3. Every group of prime order is cyclic and, therefore, abelian and normal.

Let n_3 be the number of Sylow 3-subgroups.

We know that $n_3 = 1$ or 4.

If $n_3 = 1$, the Sylow 3-subgroup is unique and, therefore, normal. This would imply the group is not simple.

Therefore, we will assume that $n_3 = 4$.

We know that these Sylow 3-subgroups must have order 3.

If there $n_3 = 4$, the intersection of these subgroups must be the identity element.

This implies that there are $4 * 2 = 8$ elements of order 3.

The remaining 4 elements of G must be belong to the Sylow 2-subgroup.

This implies that the 2^2-subgroup is unique or the 2-subgroup is unique and therefore normal.

Since G contains a normal subgroup that is neither nonoproper nor trivial, G cannot be simple.

Therefore, no group of order 12 is simple.

d.)$p^n m$ *for any prime* p *and any positve integers* n *and* m *with* $m < p$.

Let H be any Sylow p-subgroup of G.

Then $|H| = p^n$ and $[G : P] = m$, since $p > m$.

Let there be a homomorphism $\phi : G \to S_m$ such that $ker(\phi) \subseteq H$.

If $|Ker(\phi)| = p^k$, then we know that p^{n-k} divides $m!$.

This implies that $k = n$, since $p > m$.

Therefore $H = ker(\phi) \lhd G$ is a nonproper, nontrivial normal subgroup of G.

Therefore, there are no simple groups of order $p^n m$.

9.23

a.)*Find all Sylow 3-subgroups in S_4.*
A Sylow 3-group must have order 3. This means that the subgroup generator must also have order 3. Therefore, the Sylow 3-subgroups of S_4 are: $\langle(123)\rangle, \langle(124)\rangle, \langle(134)\rangle$, and $\langle(234)\rangle$.

b.)*Find all Sylow-3 subgroups in S_5.*
A Sylow 3-group must have order 3. This means that the subgroup generator must also have order 3. There are 10 subgroups of order 3: $\langle(123)\rangle, \langle(124)\rangle, \langle(125)\rangle, \langle(134)\rangle, \langle(135)\rangle, \langle(145$
and $\langle(345)\rangle$. These are all distinct. Therefore, they are all Sylow 3-subgroups.

Proof 1

Prove that a Sylow p-subgroup of G is normal if and only if it is the unique Sylow p-subgroup of G. [Hint: Recall that gHg^{-1} is always a subgroup of G.]

Suppose for the sake of contradiction that G has two Sylow normal p-subgroups, H and K.

We know from the First Sylow Theorem that $H \subseteq K$, since smaller p-subgroups are contained in the larger.

The second Sylow theorem states that any two Sylow p-subgroups are conjugates, therefore, since $|H| = |K|$, H must be a conjugate of K.

Since K was normal in G, this implies that $H = K$.

Therefore, a Sylow p-subgroup of G is normal if and only if it is the unique Sylow p-subgroup of G.

Proof 2

Recall that a group G is called simple if its only normal subgroups are G and $\{e\}$.

a.)*Show that there is no simple group of order pq, where $p < q$ and are both prime. [Hint: Show that G contains a unique Sylow subgroup for some prime.]*

Let $H \leq G$ be a Sylow p-subgroup, which is guaranteed to exist by the First Sylow Theorem. Let n_p be the total number of Sylow p-subgroups of G. Then n_p must divide q. This implies that $n_p = 1$ or $n_p = q$ and $n_p \cong_p 1$.

We know that $p > q$. Therefore it must be that $n_p = 1$. This implies that there exists a unique, nontrivial normal p-subgroup of G.

Therefore, there is no simple group of order pq, where $p < q$ and are both prime.

b.)*Show that there is no simple group of order $108 = 2^2 * 3^3$.*

$|G| = 108 = 2^2 * 3^3$. Therefore, by the Fist Sylow Theorem, there must exit Sylow 3-subgroups and 2-subgroups.

Let n_3 be the number of Sylow 3-subgroups. By Sylow's Third Theorem $n_3 = 1$ or $n_3 = 4$.

For $n_3 = 1$, G mus have a unique (and thus normal) Sylow 3-subgroup of order 3^3. Therefore, G cannot be simple.

For $n_3 = 4$, let H and K be any two distinct Sylow 3-subgroups of G (whose existence is guaranteed by the First Sylow Theorem).

We know that $|H \cap K|||H|$. This implies that $|H \cap K|$ must be $1, 3, 9$, and 27.

We have said that H and K are distinct, which implies that $|H \cap K| \neq 27$.

For $|H \cap K| \leq 3$, $|HK| = \frac{|H||K|}{|H \cap/K|} \geq \frac{3^3 * 3^3}{3} = 3^5 = 243 \geq |G| = 108$.

This implies that $|H \cap K| = 3^3$, $|HK| = 3^2 * 3^2$, and $|H| = 3^3$.

Therefore, we know that $H \cap K$ must be a normal subgroup in H and K.

The normalizer of $H \cap K$, $N(H \cap K) = \{g \in G | g(H \cap K)g^{-}1 \subseteq H \cap K\}$, which is a subgroup of G.

By construction, we know that HK must be a subset of of $N(H \cap K)$.

We know $|HK| = 81 \leq |N(H \cap K)|$ and that $|N(H \cap K)||108$, since $N(H \cap K) \triangleleft G$. This implies that $|N(H \cap K)| = 108$.

Therefore, the normalizer, $N(H \cap K) = G$.

Since $|H \cap K| = 3^2$, $|H \cap K| \triangleleft G$. Therefore, there is no simple group of order $108 = 2^2 * 3^3$.

Proof 3

Suppose that $H < G, H \neq G$ and let $S = G/H$. Let G act on S, where $\phi(g) : xH \to gxH$.

a.)*Show that if $|G|$ does not divide $[G : H]!$, then G cannot be simple.*

We will prove the contrapositive: if G is simple, then $|G||[G : H]$.

Let G act on S, where $\phi(g) : xH \to gxH$.

For $a, b \in G$, $\phi(a)\phi(b) = (ab)xH = \phi(ab)$. Therefore ϕ defines a homomorphism.

$\phi(e) = e(xH) = (ex)H = xH$. Therefore, ϕ is a nontrivial group action where $\phi : G \to Perm(S)$.

Since G is simple, the only normal subgroups are the nonproper and the trivial and ϕ must be one-to-one.

This implies that $\text{Perm}(S)$ contains a K such that $K \cong G$.

Therefore, $|G||[G : H]$.

Since the contrapositive is true, we know that if $|G|$ does not divide $[G : H]!$, then G cannot be simple.

b.)*Use (a), together with the Sylow theorems, to show that any group of order 36 cannot be simple.*

$|G| = 36 = 2^2 * 3^2$.

The First Sylow Theorem implies that there exists $H < G$.

Let $G/H = \{gH : g \in G\}$.

By definition, G/H is only a group if $H \triangleleft G$.

We will assume, for the sake of contradiction, that H is not normal in G.

Let $n_3 = [G : H] = |G|/|H|$ be the number of Sylow 3-subgroups. We know that $n_3 = 4$ or $n_3 = 1$.

However, if $n_3 = 1$, the group would not be simple. So $n_3 = 4$.

We will let $\phi : G \to Perm(S)$ be defined by $\phi(g) = gxH$.

We have previously shown that this ϕ is a hormomorphism.

$|Perm(S)| = |G : H]! = 4! = 24$.

But 36 does not divide 24.

This implies that any group of order 36 cannot be simple.