How Many Mates Can a Latin Square Have?

Marshall University Computational Sciences REU — Summer 2011 and Summer 2012

Megan Bryant — mrlebla@g.clemson.edu

James Figler — figler@live.marshall.edu

Roger Garcia — garcroge@kean.edu

Yudhishthir Singh — ysingh@crimson.ua.edu

Advisor: Carl Mummert — mummertc@marshall.edu

Introduction

The goal of this project was to investigate the number of mates that a latin square can possess using a mixture of computational and theoretical techniques. The study of latin squares is a well-established area of mathematical research,[1,3] but our question appears not to have been studied before.

We developed and ran software to perform computational searches on all squares of sizes 7 and 8. Because the number of latin squares grows very quickly as a function of the size, computational methods are required to survey the squares of a fixed size. The number of squares to consider can be reduced by looking only at squares in a canonical ("reduced") form, but the growth rate is still very fast, as shown in Table 1.

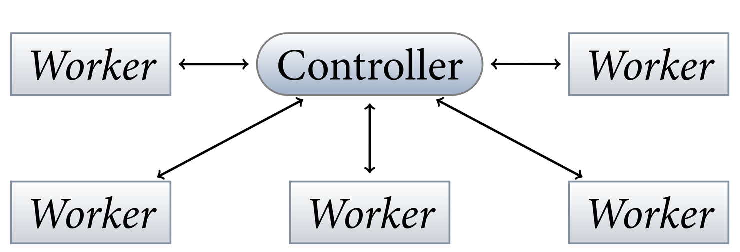
Analysis of the data from the computational search led to a general theorem on the existence of mates for certain squares whose sizes are powers of two.

Table 1.[2] **Number of reduced latin squares of sizes 4 through 11**

Size	*Number*	*Size*	*Number*
4	4	8	$\approx 54 \times 10^{10}$
5	56	9	$\approx 38 \times 10^{16}$
6	9,408	10	$\approx 76 \times 10^{23}$
7	16,942,080	11	$\approx 54 \times 10^{32}$

Software development

The first phase of the project was to develop custom software to perform exhaustive searches of all latin squares of a fixed size, counting the number of semireduced mates of each square. The algorithm divides the search space into pieces. A "controller" process sends these pieces to "worker" processes who analyze them in parallel and send the results back to the controller. The algorithm was implemented in C using the MPI interface for parallel computation.



To count the number of mates of a square S, the algorithm searches for ways to cover S with transversals. We found this to be far more efficient than simply testing whether arbitrary squares are mates of S.

Project website

A paper containing detailed results of this project, and the code that was developed, may be found at
http://science.marshall.edu/mummertc/latin2012

Latin squares, mates, and transversals

A **latin square** of size n is an $n \times n$ grid containing the numbers 1 to n such that each number appears exactly once in each row and each column. The square is **reduced** if the first row and column are in increasing order, and **semireduced** if the first row is in increasing order.

$$\begin{pmatrix} 1&2&3&4&5&6&7\\ 2&3&4&5&6&7&1\\ 3&4&5&6&7&1&2\\ 4&5&6&7&1&2&3\\ 5&6&7&1&2&3&4\\ 6&7&1&2&3&4&5\\ 7&1&2&3&4&5&6 \end{pmatrix}$$

Two latin squares of the same size are **orthogonal**, or **mates**, if every possible ordered pair occurs when the squares are superimposed.

$$\underset{\text{First square}}{\begin{pmatrix} 1&2&3\\ 2&3&1\\ 3&1&2 \end{pmatrix}} \qquad \underset{\text{Second square}}{\begin{pmatrix} 1&2&3\\ 3&1&2\\ 2&3&1 \end{pmatrix}} \qquad \underset{\text{Superimposed}}{\begin{pmatrix} (1,1)&(2,2)&(3,3)\\ (2,3)&(3,1)&(1,2)\\ (3,2)&(1,3)&(2,1) \end{pmatrix}}$$

A **transversal** of a latin square is a list of cells, one in each row and each column, which contain all the possible symbols from the square. A square has a mate if and only if it can be covered in non-overlapping transversals.

$$\begin{pmatrix} 1&\textcircled{2}&3\\ 2&3&\textcircled{1}\\ \textcircled{3}&1&2 \end{pmatrix} \qquad \begin{pmatrix} \boxed{1}&\textcircled{2}&\diamond 3\\ \diamond 2&\boxed{3}&\textcircled{1}\\ \textcircled{3}&\diamond 1&\boxed{2} \end{pmatrix} \qquad \begin{pmatrix} 1&2&3\\ 3&1&2\\ 2&3&1 \end{pmatrix}$$

A transversal (circled cells) — Covered with transversals — Mate generated by $\square = 1$, $\bigcirc = 2$, $\diamond = 3$.

The **product** operation takes two latin squares of sizes n and m and produces another latin square of size $n \cdot m$. A **power square** is a latin square that is obtained by taking a repeated product of a latin square with itself.

$$\begin{pmatrix} 1&2\\ 2&1 \end{pmatrix} \times \begin{pmatrix} 1&2&3\\ 2&3&1\\ 3&1&2 \end{pmatrix} = \left(\begin{array}{ccc|ccc} (1,1)&(1,2)&(1,3)&(2,1)&(2,2)&(2,3)\\ (1,2)&(1,3)&(1,1)&(2,2)&(2,3)&(2,1)\\ (1,3)&(1,1)&(1,2)&(2,3)&(2,1)&(2,2)\\ \hline (2,1)&(2,2)&(2,3)&(1,1)&(1,2)&(1,3)\\ (2,2)&(2,3)&(2,1)&(1,2)&(1,3)&(1,1)\\ (2,3)&(2,1)&(2,2)&(1,3)&(1,1)&(1,2) \end{array}\right) = \begin{pmatrix} 1&2&3&4&5&6\\ 2&3&1&5&6&4\\ 3&1&2&6&4&5\\ 4&5&6&1&2&3\\ 5&6&4&2&3&1\\ 6&4&5&3&1&2 \end{pmatrix}$$

Experimental results

The results of the computational searches are shown in Tables 2 and 3. The tables show frequency counts of the number of reduced latin squares by the number of semireduced mates that they possess. Each reduced latin square represents $n!(n-1)!$ distinct latin squares, while each semireduced mate represents $n!$ distinct mates. Thus the full mate counts for all latin squares can be deduced from these tables.

The computation was performed using the Marshall University computational cluster "Big Green" and an ad hoc cluster of commodity PCs. The exhaustive search for size 8 would have required about 5.5 years to complete if it was run on a single processor.

Table 2. Reduced latin squares of size 7 by number of semireduced mates

Mates	*Frequency*
0	16,765,080
1	105,840
2	52,920
3	17,640
8	210
635	120

Table 3. Reduced latin squares of size 8 by number of semireduced mates

Mates	*Frequency*	*Mates*	*Frequency*	*Mates*	*Frequency*	*Mates*	*Frequency*	*Mates*	*Frequency*	*Mates*	*Frequency*	*Mates*	*Frequency*
0	532,807,827,816	12	3,225,600	52	645,120	25	241,920	48	120,960	864	60,480	2,816	7,560
1	1,926,259,200	37	2,177,280	42	604,800	34	241,920	70	120,960	54	40,320	4,096	7,560
2	246,274,560	24	1,935,360	72	544,320	40	241,920	74	120,960	198	40,320	4,736	7,560
4	75,519,360	36	1,774,080	19	483,840	45	241,920	90	120,960	220	30,240	4,248	6,720
3	54,270,720	28	1,451,520	26	483,840	56	241,920	166	120,960	384	30,240	364	5,040
5	28,304,640	13	1,209,600	29	483,840	65	241,920	184	120,960	536	30,240	4,020	5,040
8	22,256,640	14	1,209,600	38	483,840	66	241,920	202	120,960	616	30,240	4,536	5,040
6	18,466,560	44	1,209,600	91	483,840	92	241,920	242	120,960	832	30,240	12,048	5,040
16	18,063,360	30	1,128,960	128	483,840	122	241,920	304	120,960	1,216	30,240	23,232	1,260
9	9,192,960	68	1,088,640	144	483,840	152	241,920	632	120,960	1,488	30,240	23,040	630
10	8,104,320	11	967,680	496	483,840	234	241,920	78	80,640	2,592	30,240	33,408	630
7	7,499,520	21	967,680	84	403,200	248	241,920	96	80,640	3,232	30,240	32,256	210
15	6,048,000	27	967,680	46	362,880	288	241,920	156	60,480	4,000	30,240	70,272	30
18	5,080,320	43	967,680	100	362,880	308	241,920	328	60,480	1,356	26,880		
20	4,354,560	64	967,680	50	322,560	324	241,920	332	60,480	236	15,120		
22	3,507,840	69	967,680	76	322,560	720	241,920	392	60,480	4,928	10,080		
32	3,265,920	80	846,720	352	302,400	252	201,600	528	60,480	2,496	7,560		

An existence theorem

We determined that one of the 30 latin squares of size 8 with the most semireduced mates is a power square obtained from a repeated product of the cyclic square C_2 with itself. Other computational evidence also suggests that product squares have many mates. We proved the following theorem that includes squares of arbitrarily size in its scope.

$$\begin{pmatrix} 1&2\\ 2&1 \end{pmatrix}$$

The cyclic square C_2

$$\begin{pmatrix} 1&2&3&4&5&6&7&8\\ 2&1&4&3&6&5&8&7\\ 3&4&1&2&7&8&5&6\\ 4&3&2&1&8&7&6&5\\ 5&6&7&8&1&2&3&4\\ 6&5&8&7&2&1&4&3\\ 7&8&5&6&3&4&1&2\\ 8&7&6&5&4&3&2&1 \end{pmatrix}$$

The 8×8 power square C_2^3 has 70,272 semireduced mates

Theorem

For every n, the power square C_2^n of size 2^n has at least one mate.

Future work

Computational problems

1. Perform a partial search of size 9×9 latin squares to look for squares with many mates.
2. Investigate whether latin squares that are Cayley tables of groups have many mates.

Theoretical problems

1. Analyze the mates of the 8×8 power square C_2^3 in more detail.
2. Prove a better estimate for the number of mates of a power square as a function of the size.
3. Find asymptotic bounds on the number of mates that a square can have as a function of its size.

References

[1] J. Dénes and A. D. Keedwell, *Latin squares*, Annals of Discrete Mathematics, vol. 46, North-Holland Publishing Co., Amsterdam, 1991. MR 1096296 (92h:05022)

[2] Brendan D. McKay and Ian M. Wanless, *On the number of Latin squares*, Ann. Comb. **9** (2005), no. 3, 335–344. MR 2176596 (2006f:05027)

[3] Gary L. Mullen and Carl Mummert, *Finite fields and applications*, Student Mathematical Library, vol. 41, American Mathematical Society, Providence, RI, 2007. MR 2358760 (2009b:05001)

Acknowledgments

This research was conducted during the 2011 and 2012 Marshall University REU, supported by NSF award OCI-1005117 and Marshall University. The Big Green computational cluster was supported by NSF award EPS-0918949.