CSCI 520
Homework 10

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1.) *Let X $U(0,1)$. Find the population mean, variance, skewness, and kurtosis of X.*

```
X := UniformRV(0, 1);
             [[x -> 1], [0, 1], ["Continuous", "PDF"]]

Mean(X);
                                     1
                                     -
                                     2
Variance(X);
                                     1
                                     --
                                     12
Skewness(X);
                                     0
Kurtosis(X);
                                     9
                                     -
                                     5
```

2.) *Let X (a,b). Find the population mean, variance, skewness, and kurtosis of X. Write each moment in simplified form.*

```
X := UniformRV(a, b);
           [[       1  ]                                ]
           [[x -> -----], [a, b], ["Continuous", "PDF"]]
           [[     b - a]                                ]
Mean(X);
                                1     1
                                - a + - b
                                2     2
Variance(X);
                         1   2   1       1   2
                         -- a  - - b a + -- b
                         12      6       12
Skewness(X);
          /   /
   1      |   |1  3   1    2   1  2     1  3
-------- |24 |- a  + - b a  + - b  a + - b
       3 \   \4      4        4        4
(b - a)

                                                                     3\
       /1     1  \ /1  2   1        1  2\     /1     1  \ |  (1/2)
   - 3 |- a + - b| |- a  + - b a + - b | + 2 |- a + - b| | 3
       \2     2  / \3      3        3   /     \2     2  / /
```

```
  \
  |
  |
  /
Kurtosis(X);
                                        9
                                        -
                                        5
MGF(X);
                          -exp(t a) + exp(t b)
                        - --------------------
                               (a - b) t
```

These moments may be written in the following simplified form.

$$E[X] = \frac{a}{2} + \frac{b}{2}$$
$$V[X] = \frac{a^2}{12} - \frac{ab}{6} + \frac{b^2}{12}$$
$$\text{Skewness}[X] = \frac{1}{(b-1)^3}\left(24\left(\frac{a^3}{4} + \frac{b^2 a}{4} + \frac{b^3}{4} - 3\left(\frac{a}{2} + \frac{b}{2}\right)\left(\frac{a^2}{3} + \frac{ab}{3} + \frac{b^2}{3}\right) + 2\left(\frac{a}{2} + \frac{b}{2}\right)^3\right)\sqrt{3}\right)$$
$$\text{Kurtosis}[X] = \frac{9}{5}$$
$$MGF = -\frac{e^{ta} + e^{tb}}{(a-b)t}$$

3.) *Let X Weibull(λ, k). Find the probability density function of $Y = X^2$.*

```
X := WeibullRV(lambda, k);
 [[             k  (k - 1)    /           k\]
 [[x -> k lambda  x        exp\-(lambda x) /], [0, infinity],

                         ]
   ["Continuous", "PDF"]]

g; x -> x^2;
      2
x -> x
PDF(X, g);
                 k  (k - 1)    /           k\
         k lambda  g        exp\-(lambda g) /
```

The PDF derived is

$$k\lambda^k g^{k-1} e^{-(\lambda g)^k}$$

4.) *Let X exponential(2). Use APPL's Truncate function to truncate X on the left at 1/2 and on the right at 3/2. Find the expected value of the truncated random variable as an exact expression and its floating point approximation.*

```
X := ExponentialRV(2);
   [[x -> 2 exp(-2 x)], [0, infinity], ["Continuous", "PDF"]]
ExpectedValue(Truncate(X, 1/2, 3/2));
                         -1 + 2 exp(-2)
                         --------------
                          exp(-2) - 1
evalf(ExpectedValue(Truncate(X, 1/2, 3/2)));
                          0.8434823573
```

Therefore we see that the exact expression is

$$\frac{-1 + 2e^{-2}}{e^{-2} - 1}$$

and the floating point approximation is

$$0.8434823573.$$

5.) *Let X $U(0,1)$ and Y beta$(2,1)$ be independent variables. Find the variance of the minimum X, and Y. Write a Monte Carlo Simulation in R that provides convincing numerical evidence that your APPL output is correct.*

```
X := UniformRV(0, 1);
           [[x -> 1], [0, 1], ["Continuous", "PDF"]]
Y := BetaRV(2, 1);
          [[x -> 2 x], [0, 1], ["Continuous", "PDF"]]
Variance(Minimum(X, Y));
                             43
                             ---
                             720
```

Therefore, we see that APPL provides us with a variance for the minimum of X and Y of $\frac{43}{720}$, which is approximately $0.0597\bar{2}$. Now we will conduct the following Monte Carlo Simulation in R 5 times.

```
> nrep = 10000; x = runif(nrep, 0, 1); y = rbeta(nrep, 2, 1);z = pmin(x,y, nrep); var(z);
[1] 0.06076848
> nrep = 10000; x = runif(nrep, 0, 1); y = rbeta(nrep, 2, 1);z = pmin(x,y, nrep); var(z);
[1] 0.05982626
> nrep = 10000; x = runif(nrep, 0, 1); y = rbeta(nrep, 2, 1);z = pmin(x,y, nrep); var(z);
[1] 0.05982781
> nrep = 4000000; x = runif(nrep, 0, 1); y = rbeta(nrep, 2, 1);z = pmin(x,y, nrep); var(z);
[1] 0.05968553
> nrep = 4000000; x = runif(nrep, 0, 1); y = rbeta(nrep, 2, 1);z = pmin(x,y, nrep); var(z);
[1] 0.05967373
```

The Monte Carlo simulation supports our conclusions from Maple.

6.) *Let X_1, X_2, X_3 be independent and indentically distributed $\chi^2(6)$ random variables. Find $P(X_1 + X_2 + X_3 < 20)$ using APPL. Do a web search to determine the probability distribution of $X_1 + X_2 + X_3$. In addition, write a single R statement that calculates $P(X_1 + X_2 + X_3 < 20)$.*

```
X := ChiSquareRV(6);
[[      1   2    /  1  \]                                       ]
[[x -> -- x  exp|- - x|], [0, infinity], ["Continuous", "PDF"]]
[[     16        \  2  /]                                       ]
X3 := ConvolutionIID(X, 3);
        [[        1      8    /  1  \]
        [[y -> -------- y  exp|- - y|], [0, infinity],
        [[     20643840       \  2  /]

                                   ]
          ["Continuous", "PDF"]]
                                   ]
CDF(X3, 20);
                          461843
                      1 - ------ exp(-10)
                            63
evalf(CDF(X3, 20));
                          0.6671803213
```

Therefore, we see that the probability that the sum of three random χ^2 variables is approximately 0.6671803213. A simple web search shows that the probability distribution of $X_1, X_2, and X_3$ is χ_2.

```
> for (i in 1:400000) x[i] = pchisq(sum(rchisq(3,df=6)), 14);
> mean(x);
[1] 0.6963523
> for (i in 1:400000) x[i] = pchisq(sum(rchisq(3,df=6)), 14);
> mean(x);
[1] 0.6961486
> for (i in 1:400000) x[i] = pchisq(sum(rchisq(3,df=6)), 14);
> mean(x);
[1] 0.6962944
> for (i in 1:400000) x[i] = pchisq(sum(rchisq(3,df=6)), 14);
> mean(x);
[1] 0.6958931
> for (i in 1:400000) x[i] = pchisq(sum(rchisq(3,df=6)), 14);
> mean(x);
[1] 0.696394
```

We see that our Maple code is supported by the R Monte Carlo simulation.

7.) *Let X be the product of the outcomes of four rolls of a fair die. Find $P(X < 20)$. Write a Monte Carlo simulation in R that provides convincing numerical evidence that you APPL output is correct.*

```
X1 := UniformDiscreteRV(1, 6);
 [[1  1  1  1  1  1]                                        ]
 [[-, -, -, -, -, -], [1, 2, 3, 4, 5, 6], ["Discrete", "PDF"]]
 [[6  6  6  6  6  6]                                        ]
X2 := UniformDiscreteRV(1, 6);
 [[1  1  1  1  1  1]                                        ]
 [[-, -, -, -, -, -], [1, 2, 3, 4, 5, 6], ["Discrete", "PDF"]]
 [[6  6  6  6  6  6]                                        ]
X3 := UniformDiscreteRV(1, 6);
 [[1  1  1  1  1  1]                                        ]
 [[-, -, -, -, -, -], [1, 2, 3, 4, 5, 6], ["Discrete", "PDF"]]
 [[6  6  6  6  6  6]                                        ]
X4 := UniformDiscreteRV(1, 6);
 [[1  1  1  1  1  1]                                        ]
 [[-, -, -, -, -, -], [1, 2, 3, 4, 5, 6], ["Discrete", "PDF"]]
 [[6  6  6  6  6  6]                                        ]
Z1 := Product(X1, X2);
[[1   1   1   1   1   1  1   1   1   1  1   1   1   1   1   1
[[--, --, --, --, --, -, --, --, --, -, --, --, --, --, --, --,
[[36  18  18  12  18  9  18  36  18  9  18  36  18  18  18  36

  1   1 ]
  --, --],
  18  36]

  [1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 25, 30, 36

                          ]
  ], ["Discrete", "PDF"]]
                          ]
Z2 := Product(X3, X4);
[[1   1   1   1   1   1  1   1   1   1  1   1   1   1   1   1
[[--, --, --, --, --, -, --, --, --, -, --, --, --, --, --, --,
[[36  18  18  12  18  9  18  36  18  9  18  36  18  18  18  36

  1   1 ]
  --, --],
  18  36]

  [1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 25, 30, 36

                          ]
  ], ["Discrete", "PDF"]]
                          ]
Z3 := Product(Z1, Z2);
```

```
[[ 1     1    1    5    1   1   1    1    1   1    1    19   1
[[----, ---, ---, ---, ---, --, --, ---, ---, --, ---, ----, --,
[[1296  324  324  648  324  81  81  216  108  36  108  1296  54

  1  13   1    1   1   1   1    7    1   13    1   1    5
  --, ---, ---, ---, --, --, --, ---, ---, ---, ---, --, ---,
  54  324  216  324  36  81  27  324  108  324  108  81  108

   5    5    1   1    1    1   1   1    7    5    1    1    1
  ---, ---, ---, --, ----, --, --, --, ---, ---, ---, ---, ---,
  648  108  108  54  1296  36  36  72  324  108  324  324  324

  1   1    1    1   1   1    1    7    1   1    1    1     1
  --, --, ---, ---, --, --, ---, ---, ---, --, ---, ----, ---,
  27  54  108  324  27  81  108  324  216  36  324  1296  108

  1   1    1    1   1    1    1    1   1    1    1    1    1
  --, --, ---, ---, --, ---, ---, ---, --, ---, ---, ---, ---,
  54  54  324  216  36  324  324  216  81  108  108  324  108

   1    1    1     1    1    1    1    1    1    1  ]
  ---, ---, ----, ---, ---, ---, ---, ---, ---, ----], [1, 2, 3,
  216  108  1296  324  108  324  324  216  324  1296]

  4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 25, 27, 30, 32, 36,

  40, 45, 48, 50, 54, 60, 64, 72, 75, 80, 81, 90, 96, 100, 108,

  120, 125, 128, 135, 144, 150, 160, 162, 180, 192, 200, 216,

  225, 240, 250, 256, 270, 288, 300, 320, 324, 360, 375, 384,

  400, 432, 450, 480, 500, 540, 576, 600, 625, 648, 720, 750,

                                                ]
  864, 900, 1080, 1296], ["Discrete", "PDF"]]
                                                ]
evalf(CDF(Z3, 20));
                          0.1450617284
```

Now we must perform a Monte Carlo simulation in R to support our results.
R Code:

```
function(nrep){
count = 0
for (i in 1:nrep){
```

```
x = ceiling(runif(4,0,6))
if (prod(x) <= 20) count = count + 1
}
print(count / nrep)
}
> dice_count(4000000)
[1] 0.145036
> dice_count(4000000)
[1] 0.1449205
> dice_count(4000000)
[1] 0.1448875
> dice_count(4000000)
[1] 0.1452138
> dice_count(4000000)
[1] 0.1453217
```

We see that the R output is around our Maple output of 0.1450617284 and thus supports our Maple conclusion.